Supplemental Appendix to
"Econometric Inference Using Hausman Instruments"

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Abstract

This supplement consists of two appendices. Appendix SA presents the lemmas used in proving the results from Section 3 of the paper. Appendix SB contains auxiliary lemmas utilized in the proofs of the results in Section 4.

SA Auxiliary Lemmas for Results in Section 3

Lemma SA.1 *Under Assumption 1, we have:*

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(x_{i,t}-\bar{x}_{i,\cdot})^2=\gamma^2\hat{\sigma}_c^2+\sigma_v^2(1-T^{-1})+O_p((nT)^{-1/2}).$$

Proof. Using (2), we can write

$$x_{i,t}-\bar{x}_{i,\cdot}=\gamma(c_t-\bar{c})+v_{i,t}-\bar{v}_{i,\cdot} \tag{SA.1}$$

for any $i\leq n$ and $t\leq T$. Therefore

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(x_{i,t}-\bar{x}_{i,\cdot})^2=\gamma^2T^{-1}\sum_{t\leq T}(c_t-\bar{c})^2+(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2$$
$$+2\gamma(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})(v_{i,t}-\bar{v}_{i,\cdot}). \tag{SA.2}$$

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Since $T^{-1}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2 = T^{-1}\sum_{t\leq T}v_{i,t}^2-\bar{v}_{i,\cdot}^2$, the second summation after the equality in (SA.2) can be expressed as

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2 = (nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}^2 - n^{-1}\sum_{i\leq n}\bar{v}_{i,\cdot}^2. \tag{SA.3}$$

By Assumptions 1(i, iii) and Markov's inequality,

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}^2 = \sigma_v^2 + O_p((nT)^{-1/2}). \tag{SA.4}$$

The second summation after the equality in (SA.3) can be further written as

$$n^{-1}\sum_{i\leq n}\bar{v}_{i,\cdot}^2 = (nT^2)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}^2 + 2(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}v_{i,t}v_{i,t'}. \tag{SA.5}$$

By Assumptions 1(i, iii),

$$\mathbb{E}\left[\left|(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}v_{i,t}v_{i,t'}\right|^2\right] \leq K((nT^2)^{-1}),$$

which together with (SA.4) and Markov's inequality implies that

$$n^{-1}\sum_{i\leq n}\bar{v}_{i,\cdot}^2 - T^{-1}\sigma_v^2 = O_p((nT^2)^{-1/2}). \tag{SA.6}$$

Collecting the results in (SA.3), (SA.4) and (SA.6) obtains

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(v_{i,t}-\bar{v}_{i,\cdot})^2 = \sigma_v^2(1-T^{-1}) + O_p((nT)^{-1/2}). \tag{SA.7}$$

For the third item after the equality in (SA.2), we have

$$\begin{aligned}\mathbb{E}\left[\left|(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})(v_{i,t}-\bar{v}_{i,\cdot})\right|^2\right] &= \mathbb{E}\left[\left|(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})v_{i,t}\right|^2\right] \\ &= \sigma_v^2(nT^2)^{-1}\sum_{t\leq T}\mathbb{E}\left[(c_t-\bar{c})^2\right] \leq K(nT)^{-1},\end{aligned}$$

where the second equality is by Assumptions 1(i, ii) and the law of iterated expectations, and the inequality is by Assumptions 1(ii, iii). Therefore by Markov's inequality,

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}(c_t-\bar{c})(v_{i,t}-\bar{v}_{i,\cdot}) = O_p((nT)^{-1/2}). \tag{SA.8}$$

The claim of the lemma follows from (SA.2), (SA.7) and (SA.8). ■

Lemma SA.2 *Under Assumption 1, we have:*

$$(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}x_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)=\gamma^2\hat{\sigma}_c^2+O_p((nT)^{-1/2}). \tag{SA.9}$$

Proof. Applying (SA.1) to the summand before the equality in (SA.9), in view of (3), results in:

$$\begin{aligned}(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}x_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)&=((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}x_{i,t}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)\\&=(nT)^{-1}\gamma\sum_{i\leq n}\sum_{t\leq T}x_{i,t}(c_t-\bar{c})\\&\quad+((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}x_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}),\end{aligned} \tag{SA.10}$$

where the first summation after the equality, in view of (2), can be further decomposed as

$$(nT)^{-1}\gamma\sum_{i\leq n}\sum_{t\leq T}x_{i,t}(c_t-\bar{c})=\gamma^2T^{-1}\sum_{t\leq T}c_t(c_t-\bar{c})+(nT)^{-1}\gamma\sum_{i\leq n}\sum_{t\leq T}v_{i,t}(c_t-\bar{c}). \tag{SA.11}$$

By Assumptions 1(i, ii, iii),

$$\mathbb{E}\left[\left|\sum_{i\leq n}\sum_{t\leq T}v_{i,t}(c_t-\bar{c})\right|^2\right]=\sum_{i\leq n}\sum_{t\leq T}\mathbb{E}\left[v_{i,t}^2(c_t-\bar{c})^2\right]=\sigma_v^2\sum_{i\leq n}\sum_{t\leq T}\mathbb{E}\left[(c_t-\bar{c})^2\right]\leq KnT,$$

which together with (SA.11) and Markov's inequality implies that

$$(nT)^{-1}\gamma\sum_{i\leq n}\sum_{t\leq T}x_{i,t}(c_t-\bar{c})=\gamma^2\hat{\sigma}_c^2+O_p((nT)^{-1/2}). \tag{SA.12}$$

Applying (2) to the second summation in the right hand side of (SA.10) leads to

$$\begin{aligned}((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}x_{i,t}(v_{i',t}-\bar{v}_{i',\cdot})&=\gamma((n-1)nT)^{-1}\sum_{t\leq T}c_t\sum_{i\leq n}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot})\\&\quad+((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot}).\end{aligned} \tag{SA.13}$$

Since

$$\sum_{i\leq n}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot})=\sum_{i\leq n}\sum_{i'\neq i}v_{i',t}-\sum_{i\leq n}\sum_{i'\neq i}\bar{v}_{i'\cdot}=(n-1)\sum_{i\leq n}v_{i,t}-n(n-1)\bar{v},$$

it is evident that

$$((n-1)nT)^{-1}\sum_{t\leq T}c_t\sum_{i\leq n}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot})=(nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}c_tv_{i,t}-\bar{c}\bar{v}=O_p((nT)^{-1/2}), \tag{SA.14}$$

where the second equality is by Assumptions 1(i, ii, iii) and Markov's inequality. For the second summation term on the right-hand side of (SA.13), we can write

$$\sum_{i\leq n}\sum_{t\leq T}v_{i,t}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot}) = \sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}v_{i,t}v_{i',t} - \sum_{i\leq n}\sum_{t\leq T}v_{i,t}\sum_{i'\neq i}\bar{v}_{i'\cdot}$$
$$= 2\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}v_{i,t}v_{i',t} - 2T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{v}_{i\cdot}\bar{v}_{i'\cdot}. \tag{SA.15}$$

By Assumptions 1(i, iii),

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}v_{i,t}v_{i',t}\right|^2\right] = \sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[\left|v_{i,t}v_{i',t}\right|^2\right] \leq Kn^2T$$

and

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{v}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right] = \sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[\left|\bar{v}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right] \leq Kn^2T^{-2},$$

which, along with (SA.15), show that

$$((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}v_{i,t}\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i'\cdot}) = O_p((n^2T)^{-1/2}). \tag{SA.16}$$

Collecting the results in (SA.13), (SA.14) and (SA.16) yields

$$((n-1)nT)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}x_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}) = O_p((nT)^{-1/2}). \tag{SA.17}$$

The claim of the lemma follows from (SA.10), (SA.12) and (SA.17). ■

Lemma SA.3 *Under Assumption 1, we have:*

$$\sum_{t\leq T}\sum_{i\leq n}u_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right) = \sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}) + O_p(1). \tag{SA.18}$$

Proof. In view of (SA.1), the summation term before the equality in (SA.18) can be written as:

$$\sum_{t\leq T}\sum_{i\leq n}u_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right) = (n-1)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}u_{i,t}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)$$
$$= \sum_{i\leq n}\sum_{t\leq T}\gamma u_{i,t}(c_t-\bar{c}) + (n-1)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}u_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}). \tag{SA.19}$$

We next show that

$$(n-1)^{-1}\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}u_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}) = \sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{i,t} + O_p(1), \tag{SA.20}$$

which together with (SA.19) proves the claim of the lemma. Some elementary algebra leads to

$$\sum_{i\leq n}\sum_{t\leq T}\sum_{i'\neq i}u_{i,t}(v_{i',t}-\bar{v}_{i',\cdot}) = \sum_{t\leq T}\sum_{i\leq n}\left(u_{i,t}\sum_{i'\neq i}v_{i',t}\right) - \sum_{i\leq n}\sum_{t\leq T}u_{i,t}\sum_{i'\neq i}\bar{v}_{i'\cdot}$$
$$= \sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}v_{i',t}+u_{i',t}v_{i,t}) - T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i\cdot}\bar{v}_{i'\cdot}+\bar{u}_{i'\cdot}\bar{v}_{i\cdot}). \tag{SA.21}$$

By Assumptions 1(i, iii),

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{u}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right] = \sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}[|\bar{u}_{i\cdot}|^2]\mathbb{E}[|\bar{v}_{i'\cdot}|^2] \leq Kn^2T^{-2},$$

and similarly,

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{u}_{i'\cdot}\bar{v}_{i\cdot}\right|^2\right] \leq Kn^2T^{-2},$$

which together with Markov's inequality imply that

$$(n-1)^{-1}T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i\cdot}\bar{v}_{i'\cdot}+\bar{u}_{i'\cdot}\bar{v}_{i\cdot}) = O_p(1). \tag{SA.22}$$

The desired result in (SA.20) follows from (SA.21), (SA.22) and the definition of $\varepsilon_{i,t}$ in Theorem 1. ■

Lemma SA.4 *Under Assumption 1, we have*

$$(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}) \to \tilde{\omega}_\infty Z \qquad (\mathcal{F}_0\text{-stably}),$$

where $\tilde{\omega}_\infty^2 \equiv \gamma^2\sigma_u^2\sigma_c^2 + (n_\infty-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)$ *is independent of* $Z \sim N(0,1)$.

Proof. We shall apply the stable Martingale Central Limit Theorem (MGCLT) to prove the claim of the lemma. Some notation is needed. For any $k = 1,\ldots,nT$, we define $t_k \equiv \lceil k/n\rceil$ and $i_k \equiv k - n(t_k-1)$, where $\lceil k/n\rceil$ denotes the smallest integer which is larger than or equal to k/n. Let $\mathcal{F}_{0,m}$ denote the sigma-field generated by $\{c_1,\ldots,c_{T_m}\}$. For $k=1,\ldots,nT$, let $\mathcal{F}_{k,m}$ denote the sigma field generated by $\{c_1,\ldots,c_{T_m},\{u_{i_l,t_l}\}_{l\leq k},\{v_{i_l,t_l}\}_{l\leq k}\}$. Using such notation, we can write

$$(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t}) = \sum_{k=1}^{nT}\overbrace{\frac{\gamma u_{i_k,t_k}(c_{t_k}-\bar{c})+\varepsilon_{i_k,t_k}}{(nT)^{1/2}}}^{\equiv\tilde{\eta}_k}. \tag{SA.23}$$

We first show that $\{\tilde{\eta}_k\}_{k\leq nT}$ is a martingale difference array (MDA) adapted to $\mathcal{F}_{k,m}$. By the definitions of $\mathcal{F}_{k,m}$, t_k and i_k, it is evident that $\tilde{\eta}_k$ is $\mathcal{F}_{k,m}$-measurable. It remains to show that

$$\mathbb{E}\left[\tilde{\eta}_k|\mathcal{F}_{k-1,m}\right] = 0, \quad \text{for any } k=1,\ldots,nT. \tag{SA.24}$$

For any $k = 1, \ldots, nT$, we have either: (i) $t_k = t_{k-1}$ and $i_k = i_{k-1} + 1$; or (ii) $t_k = t_{k-1} + 1$, $i_{k-1} = n$ and $i_k = 1$. In the first scenario, we can apply Assumptions 1(i, ii) to show that:

$$\mathbb{E}\left[u_{i_k,t_k}(c_{t_k} - \bar{c})|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{i_{k-1}+1,t_{k-1}}|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{i_{k-1}+1,t_{k-1}}\right] = 0,$$

and

$$\begin{aligned}\mathbb{E}\left[\varepsilon_{i_k,t_k}|\mathcal{F}_{k-1,m}\right] &= \mathbb{E}\left[(n-1)^{-1}\sum_{i'=1}^{i_k-1}(u_{i_k,t_k}v_{i',t_k} + u_{i',t_k}v_{i_k,t_k})|\mathcal{F}_{k-1,m}\right] \\ &= (n-1)^{-1}\sum_{i'=1}^{i_{k-1}}\left(\mathbb{E}\left[v_{i',t_{k-1}}u_{i_{k-1}+1,t_{k-1}}|\mathcal{F}_{k-1,m}\right] + \mathbb{E}\left[u_{i',t_{k-1}}v_{i_{k-1}+1,t_{k-1}}|\mathcal{F}_{k-1,m}\right]\right) \\ &= (n-1)^{-1}\sum_{i'=1}^{i_{k-1}}\left(v_{i',t_{k-1}}\mathbb{E}\left[u_{i_{k-1}+1,t_{k-1}}\right] + u_{i',t_{k-1}}\mathbb{E}\left[v_{i_{k-1}+1,t_{k-1}}\right]\right) = 0,\end{aligned}$$

which, together with the definition of $\tilde{\eta}_k$, shows that (SA.24) holds. Similarly, in the second scenario, we have $\varepsilon_{i_k,t_k} = 0$, and

$$\mathbb{E}\left[u_{i_k,t_k}(c_{t_k} - \bar{c})|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{1,t_{k-1}+1}|\mathcal{F}_{k-1,m}\right] = (c_{t_k} - \bar{c})\mathbb{E}\left[u_{1,t_{k-1}+1}\right] = 0,$$

which again verifies (SA.24). We next show that

$$\sum_{k=1}^{nT}\mathbb{E}\left[\left.\tilde{\eta}_k^2\right|\mathcal{F}_{k-1,m}\right] \to_p \tilde{\omega}_\infty^2, \tag{SA.25}$$

and for any $\varepsilon > 0$,

$$\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^2 I\{|\tilde{\eta}_k| > \varepsilon\}|\mathcal{F}_{k-1,m}] \to_p 0. \tag{SA.26}$$

Let $\mathcal{G}_{k,m} \equiv \bigcap_{m' \geq m}\mathcal{F}_{k,m'}$ for any $m \geq 1$ and any $k = 0, 1, \ldots, n_m T_m$. Under (SA.23)-(SA.26), we can apply the stable MGCLT, e.g., Theorem 6.1 in Häusler and Luschgy (2015), to show that

$$(nT)^{-1/2}\sum_{t \leq T}\sum_{i \leq n}(\gamma u_{i,t}(c_t - \bar{c}) + \varepsilon_{i,t}) \to \tilde{\omega}_\infty^2 Z \qquad (\mathcal{G}\text{-stably}), \tag{SA.27}$$

where $\mathcal{G}$ denotes the sigma-field generated by $\bigcup_{m=1}^\infty \mathcal{G}_{n_m T_m, m}$. Since $\mathcal{F}_0 \subset \mathcal{G}$, the claim of the lemma follows from (SA.27).[1] To verify (SA.25), we first apply Assumptions 1(i, ii) to obtain

$$(nT)\sum_{k=1}^{nT}\mathbb{E}\left[\left.\tilde{\eta}_k^2\right|\mathcal{F}_{k-1,m}\right] = \gamma^2\sum_{k=1}^{nT}\mathbb{E}\left[\left.u_{i_k,t_k}^2(c_{t_k} - \bar{c})^2\right|\mathcal{F}_{k-1,m}\right]$$

[1]We could have used the results in Kuersteiner and Prucha (2013) to establish the stable convergence, but we found it easier to work with the regularity conditions in Häusler and Luschgy (2015) in the setup of this paper.

$$
\begin{aligned}
&+ 2\gamma \sum_{k=1}^{nT} \mathbb{E}\left[(c_{t_k} - \bar{c}) u_{i_k,t_k} \varepsilon_{i_k,t_k} \middle| \mathcal{F}_{k-1,m} \right] + \sum_{k=1}^{nT} \mathbb{E}\left[\varepsilon_{i_k,t_k}^2 \middle| \mathcal{F}_{k-1,m} \right] \\
&= (nT)\gamma^2 \sigma_u^2 \hat{\sigma}_c^2 + \frac{2\gamma}{n-1} \sum_{k=1}^{nT} \sum_{i'=1}^{i_k-1} (c_{t_k} - \bar{c})(v_{i',t_k}\sigma_u^2 + u_{i',t_k}\sigma_{u,v}) \\
&+ \frac{\sigma_u^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t_k} \right)^2 + \sigma_v^2 \sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} u_{i',t_k} \right)^2}{(n-1)^2} \\
&+ \frac{2\sigma_{u,v}^2 \sum_{k=1}^{nT} \sum_{i_1'=1}^{i_k-1} \sum_{i_2'=1}^{i_k-1} u_{i_1',t_k} v_{i_2',t_k}}{(n-1)^2}. \qquad \text{(SA.28)}
\end{aligned}
$$

The second term after the second equality of (SA.28) can be written as

$$
\frac{2\gamma}{n-1} \sum_{k=1}^{nT} \sum_{i'=1}^{i_k-1} (c_{t_k} - \bar{c})(v_{i',t_k}\sigma_u^2 + u_{i',t_k}\sigma_{u,v}) = \frac{2\gamma}{n-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(\sigma_u^2 v_{i',t} + \sigma_{u,v} u_{i',t}). \qquad \text{(SA.29)}
$$

By Assumptions 1(i, ii, iii), we have

$$
\begin{aligned}
&\mathbb{E}\left[\left| \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(v_{i',t}\sigma_u^2 + u_{i',t}\sigma_{u,v}) \right|^2 \middle| \mathcal{F}_{0,m} \right] \\
&= \sum_{t \le T} (c_t - \bar{c})^2 \mathbb{E}\left[\left| \sum_{i=1}^{n-1} (n-i)(v_{i,t}\sigma_u^2 + u_{i,t}\sigma_{u,v}) \right|^2 \right] \\
&\le K \sum_{t \le T} (c_t - \bar{c})^2 \sum_{i=1}^{n-1} (n-i)^2 \le K n^3 T \hat{\sigma}_c^2
\end{aligned}
$$

and

$$
\mathbb{E}\left[\hat{\sigma}_c^2 \right] = T^{-1} \sum_{t \le T} \mathbb{E}\left[(c_t - \bar{c})^2 \right] \le T^{-1} \sum_{t \le T} \mathbb{E}\left[c_t^2 \right] \le K.
$$

Therefore, by Markov's inequality and (SA.29),

$$
\frac{2\gamma}{(n-1)nT} \sum_{k=1}^{nT} \sum_{i'=1}^{i_k-1} (c_{t_k} - \bar{c})(v_{i',t_k}\sigma_u^2 + u_{i',t_k}\sigma_{u,v}) = O_p((nT)^{-1/2}). \qquad \text{(SA.30)}
$$

We next study the third term after the second equality of (SA.28). By the definitions of i_k and t_k, we can write

$$
\begin{aligned}
&\sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t_k} \right)^2 - \frac{Tn(n-1)\sigma_v^2}{2} = \sum_{t \le T} \sum_{i=2}^{n} \left[\left(\sum_{i'=1}^{i-1} v_{i',t} \right)^2 - \sum_{i'=1}^{i-1} \sigma_v^2 \right] \\
&= \sum_{t \le T} \sum_{i=2}^{n} \left(\sum_{i'=1}^{i-1} (v_{i',t}^2 - \sigma_v^2) + 2 \sum_{i_1'=2}^{i-1} \sum_{i_2'=1}^{i_1'-1} v_{i_1',t} v_{i_2',t} \right)
\end{aligned}
$$

$$= \sum_{t\leq T}\sum_{i=1}^{n-1}(n-i)(v_{i,t}^2-\sigma_v^2) + 2\sum_{t\leq T}\sum_{i=2}^{n-1}(n-i)\sum_{i'=1}^{i-1} v_{i,t}v_{i',t}. \tag{SA.31}$$

Applying Assumptions 1(i, ii) leads to

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=1}^{n-1}(n-i)(v_{i,t}^2-\sigma_v^2)\right|^2\right] \leq \sum_{t\leq T}\sum_{i=1}^{n-1}(n-i)^2\mathbb{E}\left[v_{i,t}^4\right] \leq Kn^3T$$

and

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n-1}(n-i)\sum_{i'=1}^{i-1} v_{i,t}v_{i',t}\right|^2\right] = \sum_{t\leq T}\sum_{i=2}^{n-1}(n-i)^2\sum_{i'=1}^{i-1}\sigma_v^4 \leq Kn^4T,$$

which together with Markov's inequality and (SA.31) shows that

$$\frac{\sigma_u^2\sum_{k=1}^{nT}\left(\sum_{i'=1}^{i_k-1} v_{i',t_k}\right)^2}{n(n-1)^2T} - \frac{\sigma_u^2\sigma_v^2}{2(n-1)} = O_p((n^2T)^{-1/2}). \tag{SA.32}$$

Similar result can be established for $(nT(n-1)^2)^{-1}\sigma_v^2\sum_{k=1}^{nT}\left(\sum_{i'=1}^{i_k-1} u_{i',t_k}\right)^2$, which along with (SA.32) implies that

$$\frac{\sigma_u^2\sum_{k=1}^{nT}\left(\sum_{i'=1}^{i_k-1} v_{i',t_k}\right)^2 + \sigma_v^2\sum_{k=1}^{nT}\left(\sum_{i'=1}^{i_k-1} u_{i',t_k}\right)^2}{nT(n-1)^2} - \frac{\sigma_u^2\sigma_v^2}{n-1} = O_p((n^2T)^{-1/2}). \tag{SA.33}$$

For the last term after the second equality of (SA.28), we can write the triple summation as

$$\begin{aligned}\sum_{k=1}^{nT}\sum_{i_1'=1}^{i_k-1}\sum_{i_2'=1}^{i_k-1} u_{i_1',t_k}v_{i_2',t_k} &= \sum_{t\leq T}\sum_{i=2}^{n}\left(\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1} u_{i_1',t}v_{i_2',t}\right)\\ &= \sum_{t\leq T}\sum_{i=1}^{n-1}(n-i)u_{i,t}v_{i,t} + \sum_{t\leq T}\sum_{i=2}^{n-1}(n-i)\sum_{i'=1}^{i-1}(u_{i,t}v_{i',t}+v_{i,t}u_{i',t}).\end{aligned}$$

Applying the similar arguments for showing (SA.33) obtains

$$\frac{2\sigma_{u,v}^2\sum_{k=1}^{nT}\sum_{i_1'=1}^{i_k-1}\sum_{i_2'=1}^{i_k-1} u_{i_1',t_k}v_{i_2',t_k}}{nT(n-1)^2} - \frac{\sigma_{u,v}^2}{n-1} = O_p((nT)^{-1/2}). \tag{SA.34}$$

Collecting the results in (SA.28), (SA.30), (SA.33) and (SA.34) obtains

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_k^2\middle|\mathcal{F}_{k-1,m}\right] = \omega_{nT}^2 + O_p((nT)^{-1/2}), \tag{SA.35}$$

where $\omega_{nT}^2 \equiv \gamma^2\sigma_u^2\hat{\sigma}_c^2 + (n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)$. Since $\omega_{nT}^2 \to_p \tilde{\omega}_\infty^2$ by Assumption 1(iv), (SA.25) follows from (SA.35). We proceed to demonstrate (SA.26). Beginning with the definition of $\tilde{\eta}_k$, we have

$$\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^2 I\{|\tilde{\eta}_k|>\varepsilon\}|\mathcal{F}_{k-1,m}] \leq \varepsilon^{-2}\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^4|\mathcal{F}_{k-1,m}]$$

$$\leq \sum_{k=1}^{nT} \frac{K\gamma^4 \mathbb{E}[u_{i_k,t_k}^4 (c_{t_k} - \bar{c})^4 | \mathcal{F}_{k-1,m}]}{\varepsilon^2 (nT)^2} + \sum_{k=1}^{nT} \frac{K\mathbb{E}[\varepsilon_{i_k,t_k}^4 | \mathcal{F}_{k-1,m}]}{\varepsilon^2 (nT)^2}. \tag{SA.36}$$

By Assumptions 1(i, ii, iii), we observe that the first summation after the second inequality in (SA.36) is bounded:

$$\begin{aligned} \sum_{k=1}^{nT} \frac{\mathbb{E}[u_{i_k,t_k}^4 (c_{t_k} - \bar{c})^4 | \mathcal{F}_{k-1,m}]}{(nT)^2} &\leq K(nT)^{-2} \sum_{k=1}^{nT} (c_{t_k} - \bar{c})^4 \\ &= K(nT^2)^{-1} \sum_{t \leq T} (c_t - \bar{c})^4 = O_p((nT)^{-1}), \end{aligned} \tag{SA.37}$$

where the second equality in (SA.37) follows by

$$\mathbb{E}\left[T^{-1} \sum_{t \leq T} (c_t - \bar{c})^4\right] \leq K\mathbb{E}\left[T^{-1} \sum_{t \leq T} c_t^4 + \bar{c}^4\right] \leq KT^{-1} \sum_{t \leq T} \mathbb{E}[c_t^4] \leq K \tag{SA.38}$$

and Markov's inequality. To bound the second summation after the second inequality of (SA.36), we observe that by Assumptions 1(i, iii),

$$\begin{aligned} (n-1)^4 \sum_{k=1}^{nT} \mathbb{E}[\varepsilon_{i_k,t_k}^4 | \mathcal{F}_{k-1,m}] &\leq K \sum_{k=1}^{nT} \mathbb{E}\left[u_{i_k,t_k}^4 \left(\sum_{i'=1}^{i_k-1} v_{i',t_k}\right)^4 + v_{i_k,t_k}^4 \left(\sum_{i'=1}^{i_k-1} u_{i',t_k}\right)^4 \middle| \mathcal{F}_{k-1,m} \right] \\ &\leq K \sum_{k=1}^{nT} \left(\left(\sum_{i'=1}^{i_k-1} v_{i',t_k}\right)^4 + \left(\sum_{i'=1}^{i_k-1} u_{i',t_k}\right)^4 \right). \end{aligned} \tag{SA.39}$$

Under Assumption 1(i), we can apply Rosenthal's inequality (see, e.g., Theorem 2.12 in Hall and Heyde (1980)) to obtain:

$$\begin{aligned} \mathbb{E}\left[\sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} v_{i',t}\right)^4\right] &= \sum_{t \leq T} \sum_{i=2}^{n} \mathbb{E}\left[\left(\sum_{i'=1}^{i-1} v_{i',t}\right)^4\right] \\ &\leq K \sum_{t \leq T} \sum_{i=2}^{n} \left[\left(\sum_{i'=1}^{i-1} \mathbb{E}[v_{i',t}^2]\right)^2 + \sum_{i'=1}^{i-1} \mathbb{E}[v_{i',t}^4]\right] \\ &\leq K \sum_{t \leq T} \sum_{i=2}^{n} (i-1)^2 \leq Kn^3 T, \end{aligned} \tag{SA.40}$$

where the second inequality is by Assumption 1(iii). Similarly, we can show that

$$\mathbb{E}\left[\sum_{k=1}^{nT} \left(\sum_{i'=1}^{i_k-1} u_{i',t_k}\right)^4\right] \leq Kn^3 T,$$

which, along with (SA.39) and (SA.40) and Markov's inequality, implies that

$$(nT)^{-2} \sum_{k=1}^{nT} \mathbb{E}[\varepsilon_{i_k,t_k}^4 | \mathcal{F}_{k-1,m}] = O_p((n^3 T)^{-1}). \tag{SA.41}$$

By combining the results in (SA.36), (SA.37) and (SA.41), we derive (SA.26). ■

Lemma SA.5 *Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{i,t}^2 = \sigma_u^2(1-T^{-1}) + O_p((nT)^{-1/2}).$$

Proof. Since $\hat{u}_{i,t} = u_{i,t} - \bar{u}_{i,\cdot} - (\hat{\beta}_{iv} - \beta)(x_{i,t} - \bar{x}_{i,\cdot})$, we can write

$$\begin{aligned}(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{i,t}^2 &= (nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t} - \bar{u}_{i,\cdot})^2 \\ &\quad + (\hat{\beta}_{iv} - \beta)^2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t} - \bar{x}_{i,\cdot})^2 \\ &\quad - 2(\hat{\beta}_{iv} - \beta)(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t} - \bar{x}_{i,\cdot})(u_{i,t} - \bar{u}_{i,\cdot}). \qquad \text{(SA.42)}\end{aligned}$$

Some elementary algebra leads to

$$\begin{aligned}&(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t} - \bar{u}_{i,\cdot})^2 - \sigma_u^2(1-T^{-1}) \\ &= (nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}^2 - \sigma_u^2) - n^{-1}\sum_{i\leq n}\bar{u}_{i,\cdot}^2 + \sigma_u^2 T^{-1} \\ &= (nT)^{-1}(1-T^{-1})\sum_{i\leq n}\sum_{t\leq T}(u_{i,t}^2 - \sigma_u^2) - 2(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1} u_{i,t}u_{i,t'}. \qquad \text{(SA.43)}\end{aligned}$$

By Assumptions 1(i, iii), we have

$$\mathbb{E}\left[\left|(nT)^{-1}(1-T^{-1})\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}^2 - \sigma_u^2)\right|^2\right] \leq (nT)^{-1}\mathbb{E}[u_{i,t}^4] \leq K(nT)^{-1}$$

and

$$\mathbb{E}\left[\left|(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1} u_{i,t}u_{i,t'}\right|^2\right] = (nT^2)^{-2}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sigma_u^4 \leq K(nT^2)^{-1},$$

which together with (SA.43) and Markov's inequality shows that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t} - \bar{u}_{i,\cdot})^2 = \sigma_u^2(1-T^{-1}) + O_p((nT)^{-1/2}). \qquad \text{(SA.44)}$$

Lemma SA.1 and (8) in Theorem 1 together yield

$$(\hat{\beta}_{iv} - \beta)^2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t} - \bar{x}_{i,\cdot})^2 = O_p((nT)^{-1}). \qquad \text{(SA.45)}$$

For the third term after the equality in (SA.42), we can use the Cauchy-Schwarz inequality to get

$$
\begin{aligned}
&\left|(\hat{\beta}_{iv}-\beta)(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})(u_{i,t}-\bar{u}_{i,\cdot})\right| \\
&\leq \sqrt{(\hat{\beta}_{iv}-\beta)^2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})^2}\times\sqrt{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2} \\
&= O_p((nT)^{-1/2}), \qquad \text{(SA.46)}
\end{aligned}
$$

where the equality follows by (SA.44) and (SA.45). The claim of the lemma follows from (SA.42), (SA.44), (SA.45) and (SA.46). ■

Lemma SA.6 *Under Assumption 1, we have*

$$
(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(z_{i,t}-\bar{z}_{i,\cdot})^2=\gamma^2\hat{\sigma}_c^2+\sigma_v^2(n-1)^{-1}+O_p((nT)^{-1/2}). \qquad \text{(SA.47)}
$$

Proof. Applying (SA.1) to the term before the equality in (SA.47) leads to:

$$
\begin{aligned}
(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(z_{i,t}-\bar{z}_{i,\cdot})^2 &= (n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(x_{i',t}-\bar{x}_{i',\cdot})\right)^2 \\
&= (n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\gamma(n-1)(c_t-\bar{c})+\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2 \\
&= \gamma^2T^{-1}\sum_{t\leq T}(c_t-\bar{c})^2+(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2 \\
&\quad +2\gamma(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot}) \\
&= \gamma^2\hat{\sigma}_c^2+(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2+O_p((nT)^{-1/2}),
\end{aligned}
$$

where the last equality is by the definition of $\hat{\sigma}_c^2$, and (SA.14). It remains to show that

$$
(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2=\sigma_v^2(n-1)^{-1}+O_p((nT)^{-1/2}). \qquad \text{(SA.48)}
$$

The term before the equality in (SA.48) can be decomposed as:

$$
(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(\sum_{i'\neq i}(v_{i',t}-\bar{v}_{i',\cdot})\right)^2
$$

$$= (n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(v_{i,t}-\bar{v}_{i,\cdot}\right)^2 + 2(n-2)(n(n-1)^2T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\left(v_{i,t}-\bar{v}_{i,\cdot}\right)\left(v_{i',t}-\bar{v}_{i',\cdot}\right). \tag{SA.49}$$

Therefore, (SA.48) follows by Assumption 1(v), (SA.7) and (SA.16). ■

Lemma SA.7 *Under Assumption 1, we have (i)* $(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2 - (n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2) = O_p((n^2T)^{-1/2})$*; (ii)* $(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i',t} = O_p((nT)^{-1/2})$.

Proof. (i) By the definition of $\varepsilon_{i,t}$ and Assumption 1(i),

$$\mathbb{E}\left[\varepsilon_{i,t}^2\right] = \frac{\sum_{i'=1}^{i-1}\mathbb{E}\left[(u_{i,t}v_{i',t}+u_{i',t}v_{i,t})^2\right]}{(n-1)^2} = \frac{2\sum_{i'=1}^{i-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)}{(n-1)^2} = \frac{2(i-1)(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)}{(n-1)^2}. \tag{SA.50}$$

Therefore

$$(nT)^{-1}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2\right] = (nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\mathbb{E}\left[\varepsilon_{i,t}^2\right] = (n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2). \tag{SA.51}$$

By Assumptions 1(i, iii) and (SA.51),

$$\begin{aligned}&\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2 - (n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)\right|^2\right]\\ &= \mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\left(\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2 - \mathbb{E}\left[\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2\right]\right)\right|^2\right] \leq (nT)^{-2}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^4\right].\end{aligned} \tag{SA.52}$$

Let $\mathcal{F}_{i,t}$ denote the sigma-field generated by $\{\{u_{i',t}\}_{i'\leq i}, \{v_{i',t}\}_{i'\leq i}\}$. Then by Assumptions 1(i, ii), $\{\varepsilon_{i,t}\}_{i\leq n}$ is a MDA adapted to $\{\mathcal{F}_{i,t}\}_{i\leq n}$. By Rosenthal's inequality,

$$\mathbb{E}\left[\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^4\right] \leq K\left(\sum_{i\leq n}\mathbb{E}\left[\varepsilon_{i,t}^4\right] + \mathbb{E}\left[\left(\sum_{i\leq n}\mathbb{E}\left[\varepsilon_{i,t}^2|\mathcal{F}_{i-1,t}\right]\right)^2\right]\right). \tag{SA.53}$$

By Assumptions 1(i, iii) and Rosenthal's inequality,

$$\sum_{i\leq n}\mathbb{E}\left[\varepsilon_{i,t}^4\right] \leq \frac{K}{(n-1)^4}\sum_{i\leq n}\mathbb{E}\left[\left(\sum_{i'=1}^{i-1}v_{i',t}\right)^4 + \left(\sum_{i'=1}^{i-1}u_{i',t}\right)^4\right] \leq \frac{K}{(n-1)^4}\sum_{i\leq n}(i-1)^2 \leq Kn^{-1}. \tag{SA.54}$$

For the conditional variance of $\varepsilon_{i,t}$, we apply its definition and obtain the following upper bound:

$$\sum_{i\leq n}\mathbb{E}\left[\varepsilon_{i,t}^2|\mathcal{F}_{i-1,t}\right] \leq \frac{K}{(n-1)^2}\left[\sum_{i\leq n}\left(\sum_{i'=1}^{i-1}u_{i',t}\right)^2 + \sum_{i\leq n}\left(\sum_{i'=1}^{i-1}v_{i',t}\right)^2\right]. \tag{SA.55}$$

Since

$$\sum_{i\leq n}\left(\sum_{i'=1}^{i-1}u_{i',t}\right)^2 = \sum_{i=1}^{n-1}(n-i)u_{i,t}^2 + \sum_{i=2}^{n-1}(n-i)\sum_{i'=1}^{i-1}u_{i,t}u_{i',t}$$
$$= \frac{n(n-1)}{2}\sigma_u^2 + \sum_{i=1}^{n-1}(n-i)(u_{i,t}^2-\sigma_u^2) + \sum_{i=2}^{n-1}(n-i)\sum_{i'=1}^{i-1}u_{i,t}u_{i',t},$$

we can use Assumptions 1(i, iii) to get

$$\mathbb{E}\left[\left|\sum_{i\leq n}\left(\sum_{i'=1}^{i-1}u_{i',t}\right)^2\right|^2\right] \leq K\left(n^2(n-1)^2 + \sum_{i=1}^{n-1}(n-i)^2(i-1)\right) \leq Kn^4. \quad \text{(SA.56)}$$

Similarly, we can show that

$$\mathbb{E}\left[\left|\sum_{i\leq n}\left(\sum_{i'=1}^{i-1}v_{i',t}\right)^2\right|^2\right] \leq Kn^4,$$

which, along with (SA.55) and (SA.56), shows that

$$\mathbb{E}\left[\left(\sum_{i\leq n}\mathbb{E}\left[\varepsilon_{i,t}^2|\mathcal{F}_{i-1,t}\right]\right)^2\right] \leq K. \quad \text{(SA.57)}$$

Collecting the results in (SA.52), (SA.53), (SA.54) and (SA.57) leads to

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2 - (n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)\right|^2\right] \leq K(n^2T)^{-1},$$

which together with Markov's inequality proves the first claim of the lemma. (ii) To show the second claim of the lemma, we begin by writing

$$\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i',t} = \sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i,t} + \sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(c_t-\bar{c})(u_{i,t}\varepsilon_{i',t}+u_{i',t}\varepsilon_{i,t}). \quad \text{(SA.58)}$$

The first term after the equality in the above equation can be decomposed as

$$\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i,t} = (n-1)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\left(u_{i,t}^2\sum_{i'=1}^{i-1}v_{i',t} + u_{i,t}v_{i,t}\sum_{i'=1}^{i-1}u_{i',t}\right). \quad \text{(SA.59)}$$

By Assumptions 1(i, ii, iii), we have

$$\mathbb{E}\left[\left|((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}^2\sum_{i'=1}^{i-1}v_{i',t}\right|^2\right]$$
$$\leq K\mathbb{E}\left[\left|((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\sigma_u^2\sum_{i'=1}^{i-1}v_{i',t}\right|^2\right]$$

$$+ K\mathbb{E}\left[\left|((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}^2-\sigma_u^2)\sum_{i'=1}^{i-1}v_{i',t}\right|^2\right]$$

$$\leq K\frac{\sum_{t\leq T}\sum_{i\leq n}(n-i)^2+\sum_{t\leq T}\sum_{i\leq n}(i-1)}{((n-1)nT)^2}\leq K(nT)^{-1},$$

where $\sum_{i\leq n}\sigma_u^2\sum_{i'=1}^{i-1}v_{i',t}=\sigma_u^2\sum_{i\leq n}(n-i)v_{i,t}$ is used in deriving the second inequality. Therefore by Markov's inequality,

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}^2\sum_{i'=1}^{i-1}v_{i',t}=O_p((nT)^{-1/2}). \tag{SA.60}$$

Similarly, we can show that

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}v_{i,t}\sum_{i'=1}^{i-1}u_{i',t}=O_p((nT)^{-1/2}),$$

which along with (SA.59) and (SA.60) leads to

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i,t}=O_p((nT)^{-1/2}). \tag{SA.61}$$

To bound the second term after the equality in (SA.58), we begin by observing that by Assumptions 1(i, ii, iii),

$$\mathbb{E}\left[\left((nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})u_{i,t}\sum_{i'=1}^{i-1}\varepsilon_{i',t}\right)^2\right]\leq K(nT)^{-2}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i=2}^{n}u_{i,t}\sum_{i'=1}^{i-1}\varepsilon_{i',t}\right)^2\right]$$

$$\leq K(nT)^{-2}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[\varepsilon_{i',t}^2\right]$$

$$\leq K((n-1)nT)^{-2}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(i'-1)\leq K(nT)^{-1}.$$

Hence by Markov's inequality,

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})u_{i,t}\sum_{i'=1}^{i-1}\varepsilon_{i',t}=O_p((nT)^{-1/2}). \tag{SA.62}$$

Next note that

$$(n-1)\sum_{i=2}^{n}\varepsilon_{i,t}\sum_{i'=1}^{i-1}u_{i',t}=\sum_{i=2}^{n}v_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}u_{i_1',t}u_{i_2',t}+\sum_{i=2}^{n}u_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}v_{i_1',t}u_{i_2',t}. \tag{SA.63}$$

The first term after the equality in (SA.63) can be further written as

$$\sum_{i=2}^{n}v_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}u_{i_1',t}u_{i_2',t}=\sigma_u^2\sum_{i=2}^{n}(i-1)v_{i,t}+\sum_{i=2}^{n}v_{i,t}\sum_{i'=1}^{i-1}(u_{i',t}^2-\sigma_u^2)+2\sum_{i=2}^{n}v_{i,t}\sum_{i_1'=2}^{i-1}\sum_{i_2'=1}^{i_1'-1}u_{i_1',t}u_{i_2',t}. \tag{SA.64}$$

By Assumptions 1(i, ii, iii), we obtain the following moment bounds

$$\mathbb{E}\left[\left(((n-1)nT)^{-1}\sigma_u^2\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})(i-1)v_{i,t}\right)^2\right]\leq K(nT)^{-1},$$

$$\mathbb{E}\left[\left(((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})v_{i,t}\sum_{i'=1}^{i-1}(u_{i',t}^2-\sigma_u^2)\right)^2\right]\leq K(n^2T)^{-1},$$

$$\mathbb{E}\left[\left(((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})v_{i,t}\sum_{i_1'=2}^{i-1}\sum_{i_2'=1}^{i_1'-1}u_{i_1',t}u_{i_2',t}\right)^2\right]\leq K(nT)^{-1},$$

which together with (SA.64) and Markov's inequality implies that

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})v_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}u_{i_1',t}u_{i_2',t}=O_p((nT)^{-1/2}). \tag{SA.65}$$

By the same arguments, we can show that

$$((n-1)nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})u_{i,t}\sum_{i_1'=1}^{i-1}\sum_{i_2'=1}^{i-1}v_{i_1',t}u_{i_2',t}=O_p((nT)^{-1/2})$$

which combined with (SA.63) and (SA.65) yields

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}(c_t-\bar{c})\varepsilon_{i,t}\sum_{i'=1}^{i-1}u_{i',t}=O_p((nT)^{-1/2}). \tag{SA.66}$$

Collecting the results in (SA.62) and (SA.66) leads to

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(c_t-\bar{c})(u_{i,t}\varepsilon_{i',t}+u_{i',t}\varepsilon_{i,t})=O_p((nT)^{-1/2}). \tag{SA.67}$$

The desired result in part (ii) of the lemma follows from (SA.58), (SA.61) and (SA.67). ■

Lemma SA.8 *Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)\right)^2$$
$$=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-(\hat{\beta}_{iv}-\beta)\gamma^2n(c_t-\bar{c})^2\right)^2+O_p((nT)^{-1}).$$

Proof. Since $\hat{u}_{i,t}=u_{i,t}-\bar{u}_{i,\cdot}-(\hat{\beta}_{iv}-\beta)(x_{i,t}-\bar{x}_{i,\cdot})$, we can write

$$\sum_{i\leq n}\hat{u}_{i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}\right)=(n-1)^{-1}\sum_{i\leq n}\hat{u}_{i,t}\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)$$

$$= (n-1)^{-1}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)$$

$$-(n-1)^{-1}(\hat{\beta}_{iv}-\beta)\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right). \tag{SA.68}$$

Using (SA.1), the two terms after the equality in (SA.68) can be further written as

$$(n-1)^{-1}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right) = \sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})+I_{1,t} \tag{SA.69}$$

and

$$(n-1)^{-1}(\hat{\beta}_{iv}-\beta)\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right) = (\hat{\beta}_{iv}-\beta)\gamma^2 n(c_t-\bar{c})^2+(\hat{\beta}_{iv}-\beta)I_{2,t}, \tag{SA.70}$$

respectively, where

$$I_{1,t}\equiv(n-1)^{-1}\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}\bar{v}_{i',\cdot}-\sum_{i\leq n}u_{i,t}\sum_{i'\neq i}\bar{v}_{i',\cdot}-\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}v_{i',t}\right) \tag{SA.71}$$

and

$$I_{2,t}\equiv 2\left(\gamma(c_t-\bar{c})\left(\sum_{i\leq n}v_{i,t}-n\bar{v}\right)+(n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(v_{i,t}-\bar{v}_{i,\cdot})(v_{i',t}-\bar{v}_{i',\cdot})\right). \tag{SA.72}$$

Therefore,

$$\frac{\sum_{i\leq n}\hat{u}_{i,t}\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)}{n-1} = \sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})$$

$$-(\hat{\beta}_{iv}-\beta)\gamma^2 n(c_t-\bar{c})^2+I_{1,t}+(\hat{\beta}_{iv}-\beta)I_{2,t}. \tag{SA.73}$$

By Lemma SA.7, we can deduce that

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})\right)^2$$

$$\leq 2(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})u_{i,t}+\varepsilon_{i,t})\right)^2+2\gamma^2(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2$$

$$\leq K(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2+\left(\sum_{i\leq n}u_{i,t}\right)^2\right)+O_p(1). \tag{SA.74}$$

By Assumptions 1(i, ii, iii), we have

$$\mathbb{E}\left[(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2\right] = (nT)^{-1}\sum_{t\leq T}\mathbb{E}\left[(c_t-\bar{c})^2\right]\sum_{i\leq n}\mathbb{E}\left[\bar{u}_{i,\cdot}^2\right]\leq KT^{-1}$$

and

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}u_{i,t}\right)^2\right|\right]=(nT)^{-1}\sum_{t\leq T}\mathbb{E}\left[(c_t-\bar{c})^2\right]\sum_{i\leq n}\sigma_u^2\leq K,$$

which together with (SA.74) and Markov inequality implies that

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})\right)^2=O_p(1). \tag{SA.75}$$

By Assumptions 1(ii, iii) and Theorem 1,

$$(nT)^{-1}\sum_{t\leq T}\left((\hat{\beta}_{iv}-\beta)\gamma^2 n(c_t-\bar{c})^2\right)^2=(\hat{\beta}_{iv}-\beta)^2\gamma^4 nT^{-1}\sum_{t\leq T}(c_t-\bar{c})^4=O_p(T^{-1}). \tag{SA.76}$$

Collecting the results in (SA.75) and (SA.76) leads to

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-(\hat{\beta}_{iv}-\beta)\gamma^2 n(c_t-\bar{c})^2\right)^2=O_p(1). \tag{SA.77}$$

In view of (SA.73) and (SA.77), the claim of the lemma follows if

$$\sum_{t\leq T}I_{1,t}^2=O_p(1)\qquad\text{and}\qquad(\hat{\beta}_{iv}-\beta)^2\sum_{t\leq T}I_{2,t}^2=O_p(1). \tag{SA.78}$$

To verify the first result in (SA.78), it is sufficient to show that

$$(n-1)^{-2}\sum_{t\leq T}\left(\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\leq n}u_{i,t}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}v_{i',t}\right)^2\right)=O_p(1). \tag{SA.79}$$

By Assumptions 1(i, iii),

$$\begin{aligned}\frac{\mathbb{E}\left[\sum_{t\leq T}\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2\right]}{(n-1)^2}&=\frac{\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}\bar{v}_{i',\cdot}+\bar{u}_{i',\cdot}\bar{v}_{i,\cdot})\right)^2\right]}{(n-1)^2}\\&\leq\frac{K\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\left(\mathbb{E}\left[\bar{u}_{i,\cdot}^2\bar{v}_{i',\cdot}^2\right]+\mathbb{E}\left[\bar{u}_{i',\cdot}^2\bar{v}_{i,\cdot}^2\right]\right)}{(n-1)^2}\leq\frac{Kn}{(n-1)T},\end{aligned}$$

which together with Markov's inequality implies that

$$(n-1)^{-2}\sum_{t\leq T}\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2=O_p(T^{-1}). \tag{SA.80}$$

Similarly, we can show that

$$(n-1)^{-2}\sum_{t\leq T}\mathbb{E}\left[\left(\sum_{i\leq n}u_{i,t}\sum_{i'\neq i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\sum_{i'\neq i}v_{i',t}\right)^2\right]\leq K,$$

and hence

$$(n-1)^{-2}\sum_{t\le T}\left(\left(\sum_{i\le n}u_{i,t}\sum_{i'\ne i}\bar{v}_{i',\cdot}\right)^2+\left(\sum_{i\le n}\bar{u}_{i,\cdot}\sum_{i'\ne i}v_{i',t}\right)^2\right)=O_p(1) \tag{SA.81}$$

by Markov's inequality. The desired result in (SA.79) follows from (SA.80) and (SA.81). In view of Theorem 1, to verify the second result in (SA.78) it is sufficient to show that

$$(nT)^{-1}\sum_{t\le T}(c_t-\bar{c})^2\left(\sum_{i\le n}v_{i,t}-n\bar{v}\right)^2=O_p(1), \tag{SA.82}$$

$$(nT)^{-1}\sum_{t\le T}\left((n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(v_{i,t}-\bar{v}_{i,\cdot})(v_{i',t}-\bar{v}_{i',\cdot})\right)^2=O_p(1). \tag{SA.83}$$

By Assumptions 1(i, iii), we obtain

$$\begin{aligned}\mathbb{E}\left[(nT)^{-1}\sum_{t\le T}(c_t-\bar{c})^2\left(\sum_{i\le n}v_{i,t}-n\bar{v}\right)^2\right]&\le K(nT)^{-1}\sum_{t\le T}\mathbb{E}\left[\left(\sum_{i\le n}v_{i,t}-n\bar{v}\right)^2\right]\\&\le K(nT)^{-1}\sum_{t\le T}\mathbb{E}\left[\left(\sum_{i\le n}v_{i,t}\right)^2+n^2\bar{v}^2\right]\\&\le K(nT)^{-1}\sum_{t\le T}(n+nT^{-1})\le K,\end{aligned} \tag{SA.84}$$

which together with Markov's inequality shows (SA.82). Similarly,

$$\begin{aligned}&\mathbb{E}\left[(nT)^{-1}\sum_{t\le T}\left((n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(v_{i,t}-\bar{v}_{i,\cdot})(v_{i',t}-\bar{v}_{i',\cdot})\right)^2\right]\\&=((n-1)^2nT)^{-1}\sum_{t\le T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[((v_{i,t}-\bar{v}_{i,\cdot})(v_{i',t}-\bar{v}_{i',\cdot}))^2\right]\le Kn^{-1},\end{aligned} \tag{SA.85}$$

which together with Markov's inequality shows (SA.83). ■

Lemma SA.9 *Under Assumption 1, we have*

$$\begin{aligned}&(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-\gamma^2n(c_t-\bar{c})^2(\hat{\beta}_{iv}-\beta)\right)^2\\&=(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2+\frac{\sigma_u^2\sigma_v^2+\sigma_{u,v}^2}{n-1}+O_p((nT)^{-1/2}).\end{aligned}$$

Proof. In light of the definition of ξ_t in Lemma 2, some elementary algebra leads to:

$$
\begin{aligned}
&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})+\varepsilon_{i,t})-\gamma^2 n(c_t-\bar{c})^2(\hat{\beta}_{iv}-\beta)\right)^2\\
&=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\gamma(c_t-\bar{c})u_{i,t}-\xi_t\right)^2\\
&+2\gamma(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\leq n}(c_t-\bar{c})u_{i,t}\varepsilon_{i',t}-2\gamma\bar{u}T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\varepsilon_{i,t}\\
&-2\gamma^2(\hat{\beta}_{iv}-\beta)T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\varepsilon_{i,t}+(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\varepsilon_{i,t}\right)^2.
\end{aligned}
$$

Therefore, in view of Lemma SA.7, the claim of the lemma follows if

$$
\bar{u}T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\varepsilon_{i,t}=O_p((nT^2)^{-1/2})\text{ and }(\hat{\beta}_{iv}-\beta)T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\varepsilon_{i,t}=O_p((nT^2)^{-1/2}).\tag{SA.86}
$$

By Assumption 1(i, ii, iii) and (SA.50),

$$
\mathbb{E}\left[\bar{u}^2\right]\leq K(nT)^{-1}\quad\text{and}\quad\mathbb{E}\left[\left(T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})\varepsilon_{i,t}\right)^2\right]\leq KT^{-1},\tag{SA.87}
$$

which together with Markov's inequality shows the first result in (SA.86). Similarly,

$$
\mathbb{E}\left[\left(T^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\varepsilon_{i,t}\right)^2\right]\leq T^{-2}\sum_{t\leq T}\sum_{i\leq n}\mathbb{E}\left[(c_t-\bar{c})^4\right]\mathbb{E}\left[\varepsilon_{i,t}^2\right]\leq KT^{-1},
$$

which, along with Markov's inequality and Theorem 1, shows the second result in (SA.86). ■

Lemma SA.10 *Let $\mu_{i,t}\equiv u_{i,t}\sum_{i'=1}^{i-1}u_{i',t}$. Under Assumption 1, we have*

$$
(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2=\sigma_u^2\hat{\sigma}_c^2+2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})^2\mu_{i,t}+O_p((nT)^{-1/2})=O_p(1).
$$

Proof. By Assumptions 1(i, ii, iii),

$$
\mathbb{E}\left[(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2\right]\leq(nT)^{-1}\sum_{t\leq T}\mathbb{E}[(c_t-\bar{c})^2]\sum_{i\leq n}\mathbb{E}[u_{i,t}^2]\leq K,
$$

which together with Markov's inequality shows that

$$
(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2=O_p(1).\tag{SA.88}
$$

Since

$$\left((c_t-\bar{c})\sum_{i\leq n}u_{i,t}\right)^2=(c_t-\bar{c})^2\left(n\sigma_u^2+\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)+2\sum_{i=2}^{n}\mu_{i,t}\right),$$

we can write

$$\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})u_{i,t}\right)^2=(nT)\sigma_u^2\hat{\sigma}_c^2+\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)+2\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i=2}^{n}\mu_{i,t}. \quad \text{(SA.89)}$$

By Assumptions 1(i, ii, iii),

$$\mathbb{E}\left[\left((nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\sum_{i\leq n}(u_{i,t}^2-\sigma_u^2)\right)^2\right]\leq(nT)^{-2}\sum_{t\leq T}\mathbb{E}[(c_t-\bar{c})^4]\sum_{i\leq n}\mathbb{E}[u_{i,t}^4]\leq K(nT)^{-1},$$

which together with (SA.88), (SA.89) and Markov's inequality shows the claim of the lemma. ■

Lemma SA.11 *Under Assumption 1, we have*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})\bar{u}_{i,\cdot}+\gamma n(c_t-\bar{c})^2(\hat{\beta}_{iv}-\beta)\right)^2=O_p(T^{-1}).$$

Proof. It is evident that the claim of the lemma follows if

$$n(\hat{\beta}_{iv}-\beta)^2T^{-1}\sum_{t\leq T}(c_t-\bar{c})^4=O_p(T^{-1}), \quad \text{and} \quad \text{(SA.90)}$$

$$(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2=O_p(T^{-1}). \quad \text{(SA.91)}$$

By Assumption (iii) and Markov's inequality,

$$T^{-1}\sum_{t\leq T}(c_t-\bar{c})^4=O_p(1), \quad \text{(SA.92)}$$

which together with Theorem 1 shows (SA.90). By Assumptions (i, ii, iii),

$$\mathbb{E}\left[(nT)^{-1}\sum_{t\leq T}(c_t-\bar{c})^2\left(\sum_{i\leq n}\bar{u}_{i,\cdot}\right)^2\right]=(nT)^{-1}\sum_{t\leq T}\mathbb{E}[(c_t-\bar{c})^2]\sum_{i\leq n}\mathbb{E}[\bar{u}_{i,\cdot}^2]\leq KT^{-1},$$

which along with Markov's inequality shows (SA.91). ■

SB Auxiliary Lemmas for Results in Section 4

Lemma SB.12 *Under Assumptions 1 and 2, we have:*

$$\hat{\lambda} = \gamma \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,c} + O_p((nT)^{-1/2}) = O_p(1). \tag{SB.93}$$

Proof. Using the expression for $x_{i,t}$ in (2) and the definition of $\hat{\lambda}$, we can write

$$\hat{\lambda} = \gamma \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,c} + \hat{\Sigma}_w^{-1} (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) v_{i,t}. \tag{SB.94}$$

By Assumptions 1(i, iii) and 2(i, iv), we have

$$\mathbb{E}\left[\left|(nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) v_{i,t}\right|^2\right] = (nT)^{-2} \sum_{t \leq T} \sum_{i \leq n} \mathbb{E}\left[(w_{i,t} - \bar{w}_{i,\cdot})^2\right] \mathbb{E}\left[v_{i,t}^2\right] \leq K(nT)^{-1},$$

which, together with Markov's inequality, implies that

$$(nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} (w_{i,t} - \bar{w}_{i,\cdot}) v_{i,t} = O_p((nT)^{-1/2}). \tag{SB.95}$$

The first equality in (SB.93) follows from (SB.94) and (SB.95). To show that $\hat{\lambda}$ is stochastically bounded, we begin by applying the Cauchy-Schwarz inequality to obtain

$$\left\|\hat{\Gamma}_{w,c}\right\|^2 \leq (nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} \|w_{i,t}\|^2 \times T^{-1} \sum_{t \leq T} (c_t - \bar{c})^2. \tag{SB.96}$$

Since $(nT)^{-1} \sum_{t \leq T} \sum_{i \leq n} \|w_{i,t}\|^2 = O_p(1)$ by Assumption 2(iv) and Markov's inequality, we can use (SB.96) and Assumption 1(iii) to show that

$$\hat{\Gamma}_{w,c} = O_p(1). \tag{SB.97}$$

By Assumptions 2(ii, iv), we have

$$K^{-1} \leq \rho_{\min}(\hat{\Sigma}_w) \leq \rho_{\max}(\hat{\Sigma}_w) \leq K \tag{SB.98}$$

with probability approaching 1, where $\rho_{\min}(\hat{\Sigma}_w)$ and $\rho_{\max}(\hat{\Sigma}_w)$ denote the smallest and the largest eigenvalues of $\hat{\Sigma}_w$, respectively. Combining (SB.97) and (SB.98), we obtain

$$\hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,c} = O_p(1). \tag{SB.99}$$

The second equality in (SB.93) follows from (SB.94), (SB.95) and (SB.99). ■

Lemma SB.13 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{x}_{i,t}z_{i,t}=\gamma^2(\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+O_p((nT)^{-1/2}).$$

Proof. By the definitions of $\hat{x}_{i,t}$ and $z_{i,t}$, we can write

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{x}_{i,t}z_{i,t}=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(x_{i,t}-\bar{x}_{i,\cdot})z_{i,t}-\hat{\lambda}^\top\hat{\Gamma}_{w,z}. \tag{SB.100}$$

Given the expression for $x_{i,t}$ in (2) and the expression for $z_{i,t}$ in (3), we can decompose $\hat{\Gamma}_{w,z}$ as

$$\begin{aligned}\hat{\Gamma}_{w,z}&=(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})x_{i',t}\\&=\gamma\hat{\Gamma}_{w,c}+(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})v_{i',t}.\end{aligned} \tag{SB.101}$$

Therefore, by Lemmas SA.2 and SB.12, and using (SB.97), (SB.100) and (SB.101), the claim of the lemma follows if

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})v_{i',t}=O_p((nT)^{-1/2}). \tag{SB.102}$$

To show (SB.102), we first write

$$\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(w_{i,t}-\bar{w}_{i,\cdot})v_{i',t}=\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}v_{i',t}+w_{i',t}v_{i,t})-T\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{w}_{i\cdot}\bar{v}_{i'\cdot}+\bar{w}_{i'\cdot}\bar{v}_{i\cdot}). \tag{SB.103}$$

By Assumptions 1(i, iii) and 2(i, iv),

$$\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}w_{i,t}v_{i',t}\right|^2\right]=\sum_{t\leq T}\mathbb{E}\left[\left|\sum_{i=1}^{n-1}v_{i,t}\sum_{i'=i+1}^{n}w_{i',t}\right|^2\right]=\sigma_v^2\sum_{t\leq T}\sum_{i=1}^{n-1}\mathbb{E}\left[\left|\sum_{i'=i+1}^{n}w_{i',t}\right|^2\right]\leq Kn^3T,$$

which, together with Markov's inequality, implies that

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}w_{i,t}v_{i',t}=O_p((nT)^{-1/2}). \tag{SB.104}$$

Similarly, we can show that

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}w_{i',t}v_{i,t}=O_p((nT)^{-1/2}). \tag{SB.105}$$

Next, note that by Assumptions 1(i, iii) and 2(i, iv), and Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{w}_{i\cdot}\bar{v}_{i'\cdot}\right|^2\right]=\mathbb{E}\left[\left|\sum_{i=1}^{n-1}\bar{v}_{i\cdot}\sum_{i'=i+1}^{n}\bar{w}_{i'\cdot}\right|^2\right]=\sum_{i=1}^{n-1}\mathbb{E}\left[\bar{v}_{i\cdot}^2\right]\mathbb{E}\left[\left|\sum_{i'=i+1}^{n}\bar{w}_{i'\cdot}\right|^2\right]$$

$$\leq KT^{-1}\sum_{i=1}^{n-1}(n-i)\sum_{i'=i+1}^{n}\mathbb{E}\left[\bar{w}_{i'\cdot}^2\right]\leq Kn^3T^{-1},$$

which, together with Markov's inequality, implies that

$$(n(n-1))^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{w}_{i\cdot}\bar{v}_{i'\cdot}=O_p((nT)^{-1/2}). \tag{SB.106}$$

Similarly, we can show that

$$(n(n-1))^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\bar{w}_{i'\cdot}\bar{v}_{i\cdot}=O_p((nT)^{-1/2}),$$

which, along with (SB.103)-(SB.106), proves (SB.102). ■

Lemma SB.14 *Under Assumptions 1 and 2, we have:*

$$\hat{\beta}_{e,iv}-\beta=\frac{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c}-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i,t}-\bar{w}_{i,\cdot}))+\varepsilon_{i,t})+O_p((nT)^{-1})}{\gamma^2(\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+O_p((nT)^{-1/2})}.$$

Proof. Using the expression for $y_{i,t}$ in (14) and the definition of $\hat{\pi}$, we can write

$$\hat{\pi}=\hat{\lambda}\beta+\theta+\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}, \tag{SB.107}$$

where $\hat{\Gamma}_{w,u}\equiv(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}$. Therefore,

$$\begin{aligned}\hat{y}_{i,t}&=(x_{i,t}-\bar{x}_{i,\cdot})\beta+(u_{i,t}-\bar{u}_{i,\cdot})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}(\hat{\pi}-\theta)\\&=\hat{x}_{i,t}\beta+(u_{i,t}-\bar{u}_{i,\cdot})-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}.\end{aligned} \tag{SB.108}$$

Substituting the expression for $\hat{y}_{i,t}$ in (SB.108) into the definition of $\hat{\beta}_{e,iv}$, we obtain

$$\hat{\beta}_{e,iv}-\beta=\frac{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})z_{i,t}-\hat{\Gamma}_{w,z}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}}{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{x}_{i,t}z_{i,t}}.$$

Therefore, by Assumptions 1(iv) and 2(ii, iii, v), and Lemmas SA.3 and SB.13,

$$\hat{\beta}_{e,iv}-\beta=\frac{(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(\gamma u_{i,t}(c_t-\bar{c})+\varepsilon_{i,t})-\hat{\Gamma}_{w,z}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}+O_p((nT)^{-1})}{\gamma^2(\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+O_p((nT)^{-1/2})}. \tag{SB.109}$$

By (SB.101) and (SB.102),

$$\hat{\Gamma}_{w,z}=\gamma\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}). \tag{SB.110}$$

Applying similar arguments for showing (SB.95) yields

$$\hat{\Gamma}_{w,u}=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}w_{i,t}(u_{i,t}-\bar{u}_{i,\cdot})=O_p((nT)^{-1/2}). \tag{SB.111}$$

Combining the results from (SB.97), (SB.98), (SB.110) and (SB.111), we have

$$\hat{\Gamma}_{w,z}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u} = \gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u} + O_p((nT)^{-1}),$$

which together with (SB.109) proves the claim of the lemma. ■

Lemma SB.15 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1/2}\sum_{t\leq T}\sum_{i\leq n}(\gamma(c_t - \bar{c} - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i,t} - \bar{w}_{i,\cdot}))u_{i,t} + \varepsilon_{i,t}) \rightarrow_d \tilde{\omega}_{e,\infty}Z \qquad (\mathcal{F}_{e,0}\text{-stably}),$$

where $\tilde{\omega}_{e,\infty}^2 \equiv \gamma^2\sigma_u^2\sigma_{e,c}^2 + (n_\infty - 1)^{-1}(\sigma_u^2\sigma_v^2 + \sigma_{u,v}^2)$ *is independent of* $Z \sim N(0,1)$.

Proof. For any $k = 1, \ldots, nT$, we define $t_k = \lceil k/n \rceil$ and $i_k = k - n(t_k - 1)$. Let $\mathcal{F}_{e,0,m}$ denote the sigma-field generated by $\{\{c_t\}_{t\leq T_m}, \{w_{i,t}\}_{i\leq n_m, t\leq T_m}\}$. For $k = 1, \ldots, nT$, let $\mathcal{F}_{e,k,m}$ denote the sigma-field generated by $\{\{c_t\}_{t\leq T_m}, \{w_{i,t}\}_{i\leq n_m, t\leq T_m}, \{u_{i_l,t_l}\}_{l\leq k}, \{v_{i_l,t_l}\}_{l\leq k}\}$. Using such notation, we can write

$$\begin{aligned}(nT)^{-1/2}&\sum_{t\leq T}\sum_{i\leq n}(\gamma(c_t - \bar{c} - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i,t} - \bar{w}_{i,\cdot}))u_{i,t} + \varepsilon_{i,t}) \\ &= \sum_{k=1}^{nT}\overbrace{\frac{\gamma u_{i_k,t_k}(c_{t_k} - \bar{c} - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot})) + \varepsilon_{i_k,t_k}}{(nT)^{1/2}}}^{\equiv \tilde{\eta}_{e,k}}. \qquad \text{(SB.112)}\end{aligned}$$

By similar arguments as those used to derive (SA.24) in the proof of Lemma SA.4, we can show that $\{\tilde{\eta}_{e,k}\}_{k\leq nT}$ is an MDA adapted to $\mathcal{F}_{e,k,m}$. We next show that

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_{e,k}^2\middle|\mathcal{F}_{e,k-1,m}\right] \rightarrow_p \tilde{\omega}_{e,\infty}^2, \qquad \text{(SB.113)}$$

and for any $\varepsilon > 0$,

$$\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_{e,k}^2 I\{|\tilde{\eta}_{e,k}| > \varepsilon\}|\mathcal{F}_{e,k-1,m}] \rightarrow_p 0. \qquad \text{(SB.114)}$$

Under conditions (SB.113) and (SB.114), the claim of the lemma follows by similar arguments used in proving Lemma SA.4. By the definitions of $\tilde{\eta}_{e,k}$ and $\tilde{\eta}_k$, it follows that

$$\tilde{\eta}_{e,k} = \tilde{\eta}_k - (nT)^{-1/2}\gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i_k,t_k} - \bar{w}_{i_k,\cdot})u_{i_k,t_k}. \qquad \text{(SB.115)}$$

Therefore,

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_{e,k}^2\middle|\mathcal{F}_{e,k-1,m}\right] = \sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_k^2\middle|\mathcal{F}_{e,k-1,m}\right] + \gamma^2\sigma_u^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}$$

$$-2\gamma\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}(nT)^{-1/2}\sum_{k=1}^{nT}(w_{i_k,t_k}-\bar{w}_{i_k,\cdot})\mathbb{E}\left[\tilde{\eta}_k u_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right]. \quad \text{(SB.116)}$$

By similar arguments as those used to derive (SA.35) in the proof of Lemma SA.4, Assumptions 1(i, ii, iii) and 2(i), and the definition of $\mathcal{F}_{e,k-1,m}$, we can show that

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_k^2\middle|\mathcal{F}_{e,k-1,m}\right]=\omega_{nT}^2+O_p((nT)^{-1/2}). \quad \text{(SB.117)}$$

Since $\tilde{\eta}_k u_{i_k,t_k}=(nT)^{-1/2}(\gamma u_{i_k,t_k}^2(c_{t_k}-\bar{c})+u_{i_k,t_k}\varepsilon_{i_k,t_k})$, by Assumptions 1(i, ii, iii) and 2(i), we have

$$\begin{aligned}\mathbb{E}\left[\tilde{\eta}_k u_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right]&=(nT)^{-1/2}\gamma\sigma_u^2(c_{t_k}-\bar{c})+(nT)^{-1/2}\mathbb{E}\left[u_{i_k,t_k}\varepsilon_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right]\\&=(nT)^{-1/2}\gamma\sigma_u^2(c_{t_k}-\bar{c})+(n(n-1)^2T)^{-1/2}\sum_{i'=1}^{i_k-1}(v_{i',t_k}\sigma_u^2+u_{i',t_k}\sigma_{u,v}),\end{aligned}$$

which implies

$$\begin{aligned}&(nT)^{-1/2}\sum_{k=1}^{nT}(w_{i_k,t_k}-\bar{w}_{i_k,\cdot})\mathbb{E}\left[\tilde{\eta}_k u_{i_k,t_k}\middle|\mathcal{F}_{e,k-1,m}\right]\\&=\gamma\sigma_u^2\hat{\Gamma}_{w,c}+(n(n-1)T)^{-1}\sum_{k=1}^{nT}\sum_{i'=1}^{i_k-1}(w_{i_k,t_k}-\bar{w}_{i_k,\cdot})(v_{i',t_k}\sigma_u^2+u_{i',t_k}\sigma_{u,v})\\&=\gamma\sigma_u^2\hat{\Gamma}_{w,c}+(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}-\bar{w}_{i,\cdot})(\sigma_u^2 v_{i',t}+\sigma_{u,v}u_{i',t}). \quad \text{(SB.118)}\end{aligned}$$

By similar arguments to those used in proving (SB.102), we can show that

$$(n(n-1)T)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}-\bar{w}_{i,\cdot})(\sigma_u^2 v_{i',t}+\sigma_{u,v}u_{i',t})=O_p((nT)^{-1/2}). \quad \text{(SB.119)}$$

Combining the results from (SB.116), (SB.117), (SB.118) and (SB.119), we obtain

$$\sum_{k=1}^{nT}\mathbb{E}\left[\tilde{\eta}_{e,k}^2\middle|\mathcal{F}_{e,k-1,m}\right]=\gamma^2\sigma_u^2(\hat{\sigma}_c^2-\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+(n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)+O_p((nT)^{-1/2}). \quad \text{(SB.120)}$$

The desired result in (SB.113) follows from (SB.120) and Assumptions 1(iv) and 2(ii, iii). We now verify (SB.114). By Assumption 1(iii), (SB.115), and the Cauchy-Schwarz inequality

$$\begin{aligned}\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_{e,k}^2 I\{|\tilde{\eta}_{e,k}|>\varepsilon\}|\mathcal{F}_{e,k-1,m}]&\leq\varepsilon^{-2}\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_{e,k}^4|\mathcal{F}_{e,k-1,m}]\\&\leq K\varepsilon^{-2}\left(\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^4|\mathcal{F}_{e,k-1,m}]+(nT)^{-2}\sum_{k=1}^{nT}(\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}(w_{i_k,t_k}-\bar{w}_{i_k,\cdot}))^4\right)\end{aligned}$$

$$\leq K\varepsilon^{-2}\left(\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^4|\mathcal{F}\mathcal{F}_{e,k-1,m}]+||\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-2}\hat{\Gamma}_{w,c}||^2(nT)^{-2}\sum_{k=1}^{nT}||w_{i_k,t_k}-\bar{w}_{i_k,\cdot}||^4\right). \tag{SB.121}$$

By similar arguments as those used to derive (SA.36), (SA.37) and (SA.41) in the proof of Lemma SA.4, we have

$$\sum_{k=1}^{nT}\mathbb{E}[\tilde{\eta}_k^4|\mathcal{F}_{e,k-1,m}]=O_p((nT)^{-1}). \tag{SB.122}$$

By Assumption 2(iv) and Markov's inequality,

$$(nT)^{-2}\sum_{k=1}^{nT}||w_{i_k,t_k}-\bar{w}_{i_k,\cdot}||^4=O_p((nT)^{-1}),$$

which, together with (SB.97) and (SB.98), implies that

$$||\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-2}\hat{\Gamma}_{w,c}||^2(nT)^{-2}\sum_{k=1}^{nT}||w_{i_k,t_k}-\bar{w}_{i_k,\cdot}||^4=O_p((nT)^{-1}). \tag{SB.123}$$

Combining the results from (SB.121), (SB.122) and (SB.123), we conclude that (SB.114) holds. ■

Lemma SB.16 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{e,i,t}^2=\sigma_u^2(1-T^{-1})+O_p((nT)^{-1/2}).$$

Proof. By the definition of $\hat{u}_{e,i,t}$ and the expression for $\hat{y}_{i,t}$ in (SB.108), we can write

$$\hat{u}_{e,i,t}=(u_{i,t}-\bar{u}_{i,\cdot})-\hat{x}_{i,t}(\hat{\beta}_{e,iv}-\beta)-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}=\tilde{u}_{e,i,t}-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\phi}_w, \tag{SB.124}$$

where $\hat{\phi}_w\equiv\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,u}-(\hat{\beta}_{e,iv}-\beta)\hat{\lambda}$ and $\tilde{u}_{e,i,t}\equiv(u_{i,t}-\bar{u}_{i,\cdot})-(x_{i,t}-\bar{x}_{i,\cdot})(\hat{\beta}_{e,iv}-\beta)$. This implies

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{u}_{e,i,t}^2=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}^2+\hat{\phi}_w^{\top}\hat{\Sigma}_w\hat{\phi}_w-2(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}w_{i,t}^{\top}\hat{\phi}_w. \tag{SB.125}$$

Applying similar arguments to those used in the proof of Lemma SA.5 (replacing $\hat{\beta}_{iv}$ with $\hat{\beta}_{e,iv}$ and Theorem 1 with Theorem 3), we can show that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}^2=\sigma_u^2(1-T^{-1})+O_p((nT)^{-1/2}). \tag{SB.126}$$

By Theorem 3, Lemma SB.12, (SB.98) and (SB.111), we can deduce that

$$||\hat{\phi}_w||=O_p((nT)^{-1/2}) \quad \text{and} \quad \hat{\phi}_w^{\top}\hat{\Sigma}_w\hat{\phi}_w=O_p((nT)^{-1}). \tag{SB.127}$$

Similarly, we can show that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\tilde{u}_{e,i,t}w_{i,t}^{\top}\hat{\phi}_w = \hat{\Gamma}_{w,u}^{\top}\hat{\phi}_w - (\hat{\beta}_{e,iv}-\beta)\hat{\Gamma}_{w,x}^{\top}\hat{\phi}_w = O_p((nT)^{-1}). \tag{SB.128}$$

The claim of the lemma now follows from (SB.125) to (SB.128). ■

Lemma SB.17 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\hat{z}_{i,t}^2 = \gamma^2\hat{\sigma}_{e,c}^2 + \sigma_v^2(n-1)^{-1} + O_p((nT)^{-1/2}) \tag{SB.129}$$

where $\hat{\sigma}_{e,c}^2 \equiv \hat{\sigma}_c^2 - \hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}$.

Proof. By the definition of $\hat{z}_{i,t}$, we begin by writing

$$\begin{aligned}(nT)^{-1}&\sum_{t\leq T}\sum_{i\leq n}\hat{z}_{i,t}^2 \\ &= (nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\left(z_{i,t}-\bar{z}_{i,\cdot}-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)^2 \\ &= (nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(z_{i,t}-\bar{z}_{i,\cdot})^2 + \hat{\varphi}^{\top}\hat{\Sigma}_w\hat{\varphi} - 2\hat{\Gamma}_{w,z}^{\top}\hat{\varphi}.\end{aligned} \tag{SB.130}$$

By the definition of $\hat{\varphi}$, (SB.98), (SB.99) and (SB.110), we have

$$\hat{\varphi} \equiv \hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,z} = \gamma\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c} + O_p((nT)^{-1/2}) = O_p(1), \tag{SB.131}$$

which together with (SB.97) and (SB.98) implies that

$$\hat{\varphi}^{\top}\hat{\Sigma}_w\hat{\varphi} = \gamma^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c} + O_p((nT)^{-1/2}). \tag{SB.132}$$

Similarly, by (SB.97), (SB.98), (SB.110) and (SB.131),

$$\hat{\Gamma}_{w,z}^{\top}\hat{\varphi} = \gamma^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c} + O_p((nT)^{-1/2}). \tag{SB.133}$$

Combining the results from Lemma SA.6, (SB.130), (SB.132) and (SB.133), we establish the claim of the lemma. ■

Lemma SB.18 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)^2 = (1-T^{-1})\sigma_u^2\gamma^2\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c} + O_p(T^{-1/2}). \tag{SB.134}$$

Proof. The term on the left-hand side of (SB.134) can be expressed as:

$$
\begin{aligned}
&(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)^2 \\
&=(nT)^{-1}\sum_{t\le T}\sum_{i\le n}(u_{i,t}-\bar{u}_{i,\cdot})^2((w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi})^2 \\
&+2(nT)^{-1}\sum_{t\le T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^{\top}\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^{\top}\hat{\varphi}. \qquad \text{(SB.135)}
\end{aligned}
$$

We first analyze the double summation term on the right-hand side of (SB.135). Let $w_{i,t}^k$ and $\bar{w}_{i,t}^k$ denote the kth entries of $w_{i,t}$ and $\bar{w}_{i,t}$, respectively. By Assumptions 1(i, iii) and 2(i, iv), we can show that for any $k_1, k_2 \le d_w$:

$$
\begin{aligned}
&\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\le T}\sum_{i\le n}(u_{i,t}^2-\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right] \\
&\le (nT)^{-2}\sum_{t\le T}\sum_{i\le n}\mathbb{E}\left[u_{i,t}^4\right]\mathbb{E}[(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})^2]\le K(nT)^{-1}. \qquad \text{(SB.136)}
\end{aligned}
$$

Using Assumptions 1(i, iii), we have

$$
\begin{aligned}
\mathbb{E}\left[(\bar{u}_{i,\cdot}^2-\sigma_u^2T^{-1})^2\right] &= \mathbb{E}\left[\left(T^{-2}\sum_{t\le T}(u_{i,t}^2-\sigma_u^2)+2T^{-2}\sum_{t=2}^{T}u_{i,t}\sum_{t'=1}^{t-1}u_{i,t'}\right)^2\right] \\
&= T^{-4}\sum_{t\le T}\mathbb{E}\left[(u_{i,t}^2-\sigma_u^2)^2\right]+4T^{-4}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\mathbb{E}[u_{i,t}^2u_{i,t'}^2] \\
&\le T^{-4}\sum_{t\le T}\mathbb{E}\left[u_{i,t}^4\right]+4T^{-4}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sigma_u^4\le KT^{-2}.
\end{aligned}
$$

Combining this result with Assumptions 2(i, iv) leads to:

$$
\begin{aligned}
&\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\le T}\sum_{i\le n}(\bar{u}_{i,\cdot}^2-\sigma_u^2T^{-1})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right] \\
&=(nT)^{-2}\sum_{i\le n}\mathbb{E}\left[(\bar{u}_{i,\cdot}^2-\sigma_u^2T^{-1})^2\right]\mathbb{E}\left[\left|\sum_{t\le T}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right] \\
&\le K(nT)^{-2}\sum_{i\le n}\mathbb{E}\left[\left|T^{-1}\sum_{t\le T}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right] \\
&\le K(n^2T^3)^{-1}\sum_{i\le n}\sum_{t\le T}\mathbb{E}\left[\left|(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\le K(nT^2)^{-1}. \qquad \text{(SB.137)}
\end{aligned}
$$

Next, observe that:

$$\sum_{t\leq T}\left(u_{i,t}\sum_{t'\leq T}u_{i,t'}-\sigma_u^2\right)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})$$
$$=\sum_{t\leq T}(u_{i,t}^2-\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})$$
$$+\sum_{t=2}^{T}\sum_{t'=1}^{t-1}u_{i,t}u_{i,t'}\left((w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})+(w_{i,t'}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t'}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right). \quad \text{(SB.138)}$$

By Assumptions 1(i, iii) and 2(i, iv), we have

$$\mathbb{E}\left[\left|\sum_{t\leq T}(u_{i,t}^2-\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]$$
$$=\sum_{t\leq T}\mathbb{E}\left[(u_{i,t}^2-\sigma_u^2)^2\right]\mathbb{E}[(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})^2]\leq KT, \quad \text{(SB.139)}$$

and

$$\mathbb{E}\left[\left|\sum_{t=2}^{T}\sum_{t'=1}^{t-1}u_{i,t}u_{i,t'}\left((w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})+(w_{i,t'}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t'}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right)\right|^2\right]$$
$$=\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\mathbb{E}[u_{i,t}^2u_{i,t'}^2]\mathbb{E}[|(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})+(w_{i,t'}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t'}^{k_2}-\bar{w}_{i,\cdot}^{k_2})|^2]$$
$$\leq K\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sigma_u^4\leq KT^2. \quad \text{(SB.140)}$$

Collecting the results from (SB.138), (SB.139) and (SB.140), we conclude

$$\mathbb{E}\left[\left|\sum_{t\leq T}\left(u_{i,t}\sum_{t'\leq T}u_{i,t'}-\sigma_u^2\right)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\leq KT^2. \quad \text{(SB.141)}$$

By Assumptions 1(i) and 2(i) and using (SB.141), we obtain

$$\mathbb{E}\left[\left|(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}\bar{u}_{i,\cdot}-\sigma_u^2T^{-1})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]$$
$$=(nT^2)^{-2}\sum_{i\leq n}\mathbb{E}\left[\left|\sum_{t\leq T}\left(u_{i,t}\sum_{t'\leq T}u_{i,t'}-\sigma_u^2\right)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})\right|^2\right]\leq K(nT^2)^{-1},$$

which, along with (SB.136), (SB.137) and Markov's inequality, implies that for any $k_1,k_2\leq d_w$:

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})^2-(1-T^{-1})\sigma_u^2)(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i,t}^{k_2}-\bar{w}_{i,\cdot}^{k_2})=O_p((nT)^{-1/2}). \quad \text{(SB.142)}$$

Therefore, combining this with (SB.131) and (SB.132), it follows that

$$
\begin{aligned}
(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})^2((w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})^2 &= (1-T^{-1})\sigma_u^2\hat{\varphi}^\top\hat{\Sigma}_w\hat{\varphi}+O_p((nT)^{-1/2}) \\
&= (1-T^{-1})\sigma_u^2\gamma^2\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}). \quad \text{(SB.143)}
\end{aligned}
$$

Next, we consider the triple summation term on the right-hand side of (SB.135). Some elementary algebra yields:

$$
\begin{aligned}
&\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^\top\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^\top\hat{\varphi} \\
&=\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}u_{i',t}-u_{i,t}\bar{u}_{i',\cdot}-\bar{u}_{i,\cdot}u_{i',t}+\bar{u}_{i,\cdot}\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^\top\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^\top\hat{\varphi}. \quad \text{(SB.144)}
\end{aligned}
$$

By Assumptions 1(i, iii) and 2(i, iv), we have

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}u_{i',t}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|^2\right] \\
&=\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[u_{i,t}^2u_{i',t}^2(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})^2\right]\leq Kn^2T,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}\bar{u}_{i',\cdot}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|\right] \\
&\leq\sum_{t\leq T}\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}\bar{u}_{i',\cdot}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|\right] \\
&\leq\sum_{t\leq T}\left(\mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i,t}\bar{u}_{i',\cdot}(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})\right|^2\right]\right)^{1/2} \\
&=\sum_{t\leq T}\left(\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}\left[u_{i,t}^2\bar{u}_{i',\cdot}^2(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})^2(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})^2\right]\right)^{1/2}\leq KnT^{1/2}.
\end{aligned}
$$

Thus, by Markov's inequality:

$$
(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}u_{i',t}-u_{i,t}\bar{u}_{i',\cdot})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})=O_p(T^{-1/2}). \quad \text{(SB.145)}
$$

Similarly, we can show that

$$
(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}u_{i',t}-\bar{u}_{i,\cdot}\bar{u}_{i',\cdot})(w_{i,t}^{k_1}-\bar{w}_{i,\cdot}^{k_1})(w_{i',t}^{k_2}-\bar{w}_{i',\cdot}^{k_2})=O_p(T^{-1/2}),
$$

which, combined with (SB.131), (SB.144) and (SB.145) implies that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i,t}-\bar{w}_{i,t})^\top\hat{\varphi}(w_{i',t}-\bar{w}_{i',\cdot})^\top\hat{\varphi}=O_p(T^{-1/2}). \tag{SB.146}$$

The claim of the lemma follows from (SB.135), (SB.143) and (SB.146). ■

Lemma SB.19 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi}\right)$$
$$=(1-T^{-1})\sigma_u^2\gamma\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p(T^{-1/2}). \tag{SB.147}$$

Proof. Some elementary algebra yields:

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})\right)$$
$$=(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})^2(w_{i,t}-\bar{w}_{i,\cdot})$$
$$+(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i',t}-\bar{w}_{i',\cdot}). \tag{SB.148}$$

By Assumptions 1(i, ii, iii) and 2(i, iv), we can use similar arguments to those for proving (SB.142) to show that

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})^2-(1-T^{-1})\sigma_u^2)(c_t-\bar{c})(w_{i,t}-\bar{w}_{i,\cdot})=O_p((nT)^{-1/2}).$$

Combining this with (SB.97), (SB.131) and (SB.132) leads to

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})^2(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi}$$
$$=(1-T^{-1})\sigma_u^2\hat{\Gamma}_{w,c}^\top\hat{\varphi}+O_p((nT)^{-1/2})=(1-T^{-1})\sigma_u^2\gamma\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p((nT)^{-1/2}). \tag{SB.149}$$

The triple summation on the right-hand side of (SB.148) can be written as

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\neq i}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i',t}-\bar{w}_{i',\cdot})$$
$$=(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(c_t-\bar{c})(u_{i,t}-\bar{u}_{i,\cdot})(u_{i',t}-\bar{u}_{i',\cdot})(w_{i',t}-\bar{w}_{i',\cdot})$$

$$+ (nT)^{-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(u_{i',t} - \bar{u}_{i',\cdot})(u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot}). \tag{SB.150}$$

Using similar arguments to those used for proving (SB.146), we can show that

$$(nT)^{-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(u_{i,t} - \bar{u}_{i,\cdot})(u_{i',t} - \bar{u}_{i',\cdot})(w_{i',t} - \bar{w}_{i',\cdot})^\top \hat{\varphi} = O_p(T^{-1/2}),$$

and

$$(nT)^{-1} \sum_{t \le T} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (c_t - \bar{c})(u_{i',t} - \bar{u}_{i',\cdot})(u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} = O_p(T^{-1/2}),$$

which, together with (SB.131) and (SB.150), implies that

$$(nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (c_t - \bar{c})(u_{i,t} - \bar{u}_{i,\cdot})(u_{i',t} - \bar{u}_{i',\cdot})(w_{i',t} - \bar{w}_{i',\cdot})^\top \hat{\varphi} = O_p(T^{-1/2}). \tag{SB.151}$$

The claim of the lemma follows from (SB.148), (SB.149) and (SB.151) shows the claim of the lemma. ■

Lemma SB.20 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} (u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot}) \right) \left(\sum_{i \le n} \varepsilon_{i,t} \right) = O_p(T^{-1/2}).$$

Proof. Without loss of generality, we assume that $w_{i,t}$ is a scalar throughout the proof of this lemma. If $w_{i,t}$ is a vector, the proof can be applied componentwise. We begin the proof by writing:

$$\begin{aligned}
&(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} (u_{i,t} - \bar{u}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot}) \right) \left(\sum_{i \le n} \varepsilon_{i,t} \right) \\
&= (nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) u_{i,t} \varepsilon_{i',t} - (nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) \bar{u}_{i,\cdot} \varepsilon_{i',t}.
\end{aligned}$$

Applying similar arguments to those used for proving Lemma SA.7(ii) with $c_t - \bar{c}$ replaced by $w_{i,t} - \bar{w}_{i,\cdot}$, we can show that under Assumptions 1(i, iii) and 2(i),

$$(nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) u_{i,t} \varepsilon_{i',t} = O_p((nT)^{-1/2}).$$

Therefore, the claim of the lemma follows if

$$(nT)^{-1} \sum_{t \le T} \sum_{i \le n} \sum_{i' \le n} (w_{i,t} - \bar{w}_{i,\cdot}) \bar{u}_{i,\cdot} \varepsilon_{i',t} = O_p((nT)^{-1/2}). \tag{SB.152}$$

To demonstrate the result in (SB.152), we first decompose the triple summation term on its left-hand side as follows

$$\sum_{t\leq T}\sum_{i\leq n}\sum_{i'\leq n}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot}\varepsilon_{i',t}=\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot}\varepsilon_{i,t}$$
$$+\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t}+\bar{u}_{i',\cdot}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}). \tag{SB.153}$$

The first term on the right-hand side can be further decomposed as:

$$\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot}\varepsilon_{i,t}=T^{-1}\sum_{t\leq T}\sum_{i\leq n}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}$$
$$+T^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}(\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})u_{i,t}+\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t'}). \tag{SB.154}$$

Using similar arguments as those employed in deriving (SA.61) with $c_t-\bar{c}$ replaced by $w_{i,t}-\bar{w}_{i,\cdot}$, we can show that under Assumptions 1(i, iii) and 2(i),

$$(nT^2)^{-1}\sum_{t\leq T}\sum_{i\leq n}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}\varepsilon_{i,t}=O_p((nT^3)^{-1/2}). \tag{SB.155}$$

For any $i_2>i_1$ and any $t_k\neq t'_k$ for $k=1,2$, we have

$$\begin{aligned}&\mathbb{E}[\varepsilon_{i_1,t'_1}(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})u_{i_1,t_1}\varepsilon_{i_2,t'_2}(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})u_{i_2,t_2}]\\&=\mathbb{E}[(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i_1,t'_1}u_{i_1,t_1}\varepsilon_{i_2,t'_2}u_{i_2,t_2}]\\&=\mathbb{E}[(w_{i_1,t'_1}-\bar{w}_{i_1,\cdot})(w_{i_2,t'_2}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i_1,t'_1}u_{i_1,t_1}\varepsilon_{i_2,t'_2}]\mathbb{E}[u_{i_2,t_2}]=0,\end{aligned}$$

where the first equality is by Assumption 2(i), and the subsequent equalities follow from Assumptions 1(i, iii) and 2(iv). Therefore,

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})u_{i,t}\right|^2\right]&=\sum_{i\leq n}\mathbb{E}\left[\left|\sum_{t=2}^{T}u_{i,t}\sum_{t'=1}^{t-1}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})\right|^2\right]\\&=K\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\mathbb{E}[(w_{i,t'}-\bar{w}_{i,\cdot})^2]\mathbb{E}[\varepsilon_{i,t'}^2]\mathbb{E}[u_{i,t}^2]\\&\leq K(n-1)^{-2}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i\leq n}(i-1)\leq KT^2,\end{aligned}$$

which together with Markov's inequality shows that

$$(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})u_{i,t}=O_p((nT)^{-1}). \tag{SB.156}$$

Similarly,

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t'}\right|^2\right] &= \sum_{i\leq n}\mathbb{E}\left[\left|\sum_{t=1}^{T-1}\sum_{t'=t+1}^{T}u_{i,t}\varepsilon_{i,t'}(w_{i,t'}-\bar{w}_{i,\cdot})\right|^2\right]\\ &= \sum_{i\leq n}\sum_{t=1}^{T-1}\sum_{t'=t+1}^{T}\mathbb{E}[(w_{i,t'}-\bar{w}_{i,\cdot})^2]\mathbb{E}[\varepsilon_{i,t'}^2]\mathbb{E}[u_{i,t}^2]\\ &\leq K(n-1)^{-2}\sum_{t=1}^{T-1}\sum_{t'=t+1}^{T}\sum_{i\leq n}(i-1)\leq KT^2.\end{aligned}$$

Thus, by Markov's inequality

$$(nT^2)^{-1}\sum_{i\leq n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})u_{i,t'}=O_p((nT)^{-1}).$$

Combining this with (SB.154), (SB.155) and (SB.156) yields:

$$(nT)^{-1}\sum_{t\leq T}\sum_{i\leq n}\varepsilon_{i,t}(w_{i,t}-\bar{w}_{i,\cdot})\bar{u}_{i,\cdot}=O_p((nT)^{-1}). \tag{SB.157}$$

Next, we examine the second term after the equality in (SB.153), which can be decomposed as

$$\begin{aligned}&\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t}+\bar{u}_{i',\cdot}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t})\\ &= T^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}((w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}\varepsilon_{i',t}+(w_{i',t}-\bar{w}_{i',\cdot})u_{i',t}\varepsilon_{i,t})\\ &+T^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i,t}(w_{i,t'}-\bar{w}_{i,\cdot})\varepsilon_{i',t'}+u_{i,t'}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t})\\ &+T^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i',t}(w_{i',t'}-\bar{w}_{i',\cdot})\varepsilon_{i,t'}+u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}).\end{aligned} \tag{SB.158}$$

Using similar arguments as those used to derive (SA.67) with $c_t-\bar{c}$ replaced by $w_{i,t}-\bar{w}_{i,\cdot}$, we can show that

$$(nT^2)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}((w_{i,t}-\bar{w}_{i,\cdot})u_{i,t}\varepsilon_{i',t}+(w_{i',t}-\bar{w}_{i',\cdot})u_{i',t}\varepsilon_{i,t})=O_p((nT^3)^{-1/2}). \tag{SB.159}$$

For any $i_2>i_1$, any $i_k'<i_k$, any t_k and t_k' for $k=1,2$, we have

$$\begin{aligned}&\mathbb{E}[u_{i_1,t_1}(w_{i_1,t_1'}-\bar{w}_{i_1,\cdot})\varepsilon_{i_1',t_1'}u_{i_2,t_2}(w_{i_2,t_2'}-\bar{w}_{i_2,\cdot})\varepsilon_{i_2',t_2'}]\\ &=\mathbb{E}[(w_{i_1,t_1'}-\bar{w}_{i_1,\cdot})(w_{i_2,t_2'}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i_1',t_1'}u_{i_1,t_1}\varepsilon_{i_2',t_2'}u_{i_2,t_2}]\\ &=\mathbb{E}[(w_{i_1,t_1'}-\bar{w}_{i_1,\cdot})(w_{i_2,t_2'}-\bar{w}_{i_2,\cdot})]\mathbb{E}[\varepsilon_{i_1',t_1'}u_{i_1,t_1}\varepsilon_{i_2',t_2'}]\mathbb{E}[u_{i_2,t_2}]=0,\end{aligned} \tag{SB.160}$$

where the first equality is by Assumption 2(i), and the subsequent equalities follow from Assumptions 1(i, iii) and 2(iv). Additionally, for any $t_2 > t_1$, and for any $i'_k < i$ and $t'_k < t_k$ for $k = 1, 2$, we have

$$\begin{aligned} &\mathbb{E}[u_{i,t_1}(w_{i,t'_1} - \bar{w}_{i,\cdot})\varepsilon_{i'_1,t'_1} u_{i,t_2}(w_{i,t'_2} - \bar{w}_{i,\cdot})\varepsilon_{i'_2,t'_2}] \\ &= \mathbb{E}[(w_{i,t'_1} - \bar{w}_{i,\cdot})(w_{i,t'_2} - \bar{w}_{i,\cdot})]\mathbb{E}[\varepsilon_{i'_1,t'_1} u_{i,t_1}\varepsilon_{i'_2,t'_2}]\mathbb{E}[u_{i,t_2}] = 0. \end{aligned} \tag{SB.161}$$

By (SB.160) and (SB.161), we have

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{i=2}^{n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i'=1}^{i-1} u_{i,t}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'}\right|^2\right] &= \sum_{i=2}^{n}\sum_{t=2}^{T}\mathbb{E}\left[u_{i,t}^2\left|\sum_{t'=1}^{t-1}\sum_{i'=1}^{i-1}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'}\right|^2\right] \\ &= \sigma_u^2\sum_{i=2}^{n}\sum_{t=2}^{T}\mathbb{E}\left[\left|\sum_{t'=1}^{t-1}(w_{i,t'} - \bar{w}_{i,\cdot})\sum_{i'=1}^{i-1}\varepsilon_{i',t'}\right|^2\right] \\ &= \sigma_u^2\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\mathbb{E}[(w_{i,t'} - \bar{w}_{i,\cdot})^2]\mathbb{E}\left[\left|\sum_{i'=1}^{i-1}\varepsilon_{i',t'}\right|^2\right] \\ &= \sigma_u^2\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}\mathbb{E}[(w_{i,t'} - \bar{w}_{i,\cdot})^2]\mathbb{E}[\varepsilon_{i',t'}^2] \\ &\leq K(n-1)^{-2}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(i'-1) \leq KnT^2, \end{aligned} \tag{SB.162}$$

where the second, the third, and the fourth equalities are by Assumptions 1(i) and 2(i), and the first inequality follows from Assumptions 1(i, iii) and 2(iv). Thus, by Markov's inequality

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1} u_{i,t}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'} = O_p((nT^2)^{-1/2}). \tag{SB.163}$$

For any $t_2 > t_1$, and for any $i'_k < i$ and $t'_k < t_k$ for $k = 1, 2$, we have

$$\begin{aligned} &\mathbb{E}[u_{i,t'_1}(w_{i,t_1} - \bar{w}_{i,\cdot})\varepsilon_{i'_1,t_1} u_{i,t'_2}(w_{i,t_2} - \bar{w}_{i,\cdot})\varepsilon_{i'_2,t_2}] \\ &= \mathbb{E}[(w_{i,t_1} - \bar{w}_{i,\cdot})(w_{i,t_2} - \bar{w}_{i,\cdot})]\mathbb{E}[u_{i,t'_1}\varepsilon_{i'_1,t_1} u_{i,t'_2}\varepsilon_{i'_2,t_2}] \\ &= \mathbb{E}[(w_{i,t_1} - \bar{w}_{i,\cdot})(w_{i,t_2} - \bar{w}_{i,\cdot})]\mathbb{E}[u_{i,t'_1}\varepsilon_{i'_1,t_1} u_{i,t'_2}]\mathbb{E}[\varepsilon_{i'_2,t_2}] = 0, \end{aligned} \tag{SB.164}$$

where the first equality is by Assumption 2(i), and the subsequent equalities follow from Assumptions 1(i, iii) and 2(iv). By (SB.160) and (SB.164), and using similar arguments as those for deriving (SB.162), we obtain:

$$\mathbb{E}\left[\left|\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1} u_{i,t'}(w_{i,t} - \bar{w}_{i,\cdot})\varepsilon_{i',t}\right|^2\right] = \sum_{i=2}^{n}\sum_{t=2}^{T}\mathbb{E}\left[(w_{i,t} - \bar{w}_{i,\cdot})^2\left|\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1} u_{i,t'}\varepsilon_{i',t}\right|^2\right]$$

$$= \sum_{i=2}^{n} \sum_{t=2}^{T} \mathbb{E}[(w_{i,t} - \bar{w}_{i,\cdot})^2] \mathbb{E}\left[\left|\sum_{t'=1}^{t-1} u_{i,t'} \sum_{i'=1}^{i-1} \varepsilon_{i',t}\right|^2\right]$$

$$= \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{i'=1}^{i-1} \sum_{t'=1}^{t-1} \mathbb{E}[(w_{i,t} - \bar{w}_{i,\cdot})^2] \mathbb{E}[u_{i,t'}^2] \mathbb{E}[\varepsilon_{i',t}^2]$$

$$\leq K(n-1)^2 \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i'=1}^{i-1} (i'-1) \leq KnT^2.$$

Thus, by Markov's inequality

$$(nT^2)^{-1} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} u_{i,t'}(w_{i,t} - \bar{w}_{i,\cdot})\varepsilon_{i',t} = O_p((nT^2)^{-1/2}),$$

which, along with (SB.163), shows that

$$(nT^2)^{-1} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (u_{i,t}(w_{i,t'} - \bar{w}_{i,\cdot})\varepsilon_{i',t'} + u_{i,t'}(w_{i,t} - \bar{w}_{i,\cdot})\varepsilon_{i',t}) = O_p((nT^2)^{-1/2}). \tag{SB.165}$$

For any $t_2 > t_1$, any $t'_k < t_k$, any i_k and i'_k for $k = 1, 2$, we have

$$\begin{aligned} &\mathbb{E}[u_{i'_1,t_1}(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t'_1} u_{i'_2,t_2}(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t'_2}] \\ &= \mathbb{E}[(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})] \mathbb{E}[u_{i'_1,t_1}\varepsilon_{i_1,t'_1}\varepsilon_{i_2,t'_2}] \mathbb{E}[u_{i'_2,t_2}] = 0, \end{aligned} \tag{SB.166}$$

where the first equality follows from Assumptions 1(i, iii) and 2(iv). Additionally, for any $i_2 > i_1$, and for any $t'_k < t$ and $i'_k < i_k$ for $k = 1, 2$, we have

$$\begin{aligned} &\mathbb{E}[u_{i'_1,t}(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t'_1} u_{i'_2,t}(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t'_2}] \\ &= \mathbb{E}[(w_{i'_1,t'_1} - \bar{w}_{i'_1,\cdot})(w_{i'_2,t'_2} - \bar{w}_{i'_2,\cdot})] \mathbb{E}[u_{i'_1,t}\varepsilon_{i_1,t'_1} u_{i'_2,t}] \mathbb{E}[\varepsilon_{i_2,t'_2}] = 0. \end{aligned} \tag{SB.167}$$

By (SB.166) and (SB.167),

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} u_{i',t}(w_{i',t'} - \bar{w}_{i',\cdot})\varepsilon_{i,t'}\right|^2\right] &= \sum_{t=2}^{T} \sum_{i=2}^{n} \mathbb{E}\left[\left|\sum_{i'=1}^{i-1} u_{i',t} \sum_{t'=1}^{t-1} (w_{i',t'} - \bar{w}_{i',\cdot})\varepsilon_{i,t'}\right|^2\right] \\ &= \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{i'=1}^{i-1} \mathbb{E}[u_{i',t}^2] \mathbb{E}\left[\left|\sum_{t'=1}^{t-1} (w_{i',t'} - \bar{w}_{i',\cdot})\varepsilon_{i,t'}\right|^2\right] \\ &= \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{i'=1}^{i-1} \sum_{t'=1}^{t-1} \mathbb{E}[u_{i',t}^2] \mathbb{E}[(w_{i',t'} - \bar{w}_{i',\cdot})^2] \mathbb{E}[\varepsilon_{i,t'}^2] \\ &\leq K(n-1)^2 \sum_{i=2}^{n} \sum_{t=2}^{T} \sum_{t'=1}^{t-1} \sum_{i'=1}^{i-1} (i'-1) \leq KnT^2, \end{aligned}$$

which, together with Markov's inequality, shows that

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i',t}(w_{i',t'}-\bar{w}_{i',\cdot})\varepsilon_{i,t'}=O_p((nT^2)^{-1/2}). \tag{SB.168}$$

For any $t_2>t_1$, any i_k and i'_k, any $t'_k<t_k$ for $k=1,2$, we have

$$\begin{aligned}&\mathbb{E}[u_{i'_1,t'_1}(w_{i'_1,t_1}-\bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t_1}u_{i'_2,t'_2}(w_{i'_2,t_2}-\bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t_2}]\\&=\mathbb{E}[(w_{i'_1,t_1}-\bar{w}_{i'_1,\cdot})(w_{i'_2,t_2}-\bar{w}_{i'_2,\cdot})]\mathbb{E}[u_{i'_1,t'_1}\varepsilon_{i_1,t_1}u_{i'_2,t'_2}]\mathbb{E}[\varepsilon_{i_2,t_2}]=0,\end{aligned} \tag{SB.169}$$

where the first equality follows from Assumptions 1(i, iii) and 2(iv). Additionally for any $i_2>i_1$, and for any $t'_k<t$ and $i'_k<i_k$ for $k=1,2$, we have

$$\begin{aligned}&\mathbb{E}[u_{i'_1,t'_1}(w_{i'_1,t}-\bar{w}_{i'_1,\cdot})\varepsilon_{i_1,t}u_{i'_2,t'_2}(w_{i'_2,t}-\bar{w}_{i'_2,\cdot})\varepsilon_{i_2,t}]\\&=\mathbb{E}[(w_{i'_1,t}-\bar{w}_{i'_1,\cdot})(w_{i'_2,t}-\bar{w}_{i'_2,\cdot})]\mathbb{E}[u_{i'_1,t'_1}\varepsilon_{i_1,t}u_{i'_2,t'_2}]\mathbb{E}[\varepsilon_{i_2,t}]=0.\end{aligned} \tag{SB.170}$$

By (SB.169) and (SB.170),

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}\right|^2\right]&=\sum_{t=2}^{T}\sum_{i=2}^{n}\mathbb{E}\left[\varepsilon_{i,t}^2\left|\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\right|^2\right]\\&=\sum_{t=2}^{T}\sum_{i=2}^{n}\mathbb{E}[\varepsilon_{i,t}^2]\mathbb{E}\left[\left|\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\right|^2\right]\\&=\sum_{i=2}^{n}\sum_{t=2}^{T}\sum_{i'=1}^{i-1}\sum_{t'=1}^{t-1}\mathbb{E}[\varepsilon_{i,t}^2]\mathbb{E}[u_{i',t'}^2]\mathbb{E}[(w_{i',t}-\bar{w}_{i',\cdot})^2]\\&\leq K(n-1)^2\sum_{i=2}^{n}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i'=1}^{i-1}(i'-1)\leq KnT^2.\end{aligned}$$

Thus, by Markov's inequality

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t}=O_p((nT^2)^{-1/2}),$$

which along with (SB.168) shows that

$$(nT^2)^{-1}\sum_{t=2}^{T}\sum_{t'=1}^{t-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(u_{i',t}(w_{i',t'}-\bar{w}_{i',\cdot})\varepsilon_{i,t'}+u_{i',t'}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t})=O_p((nT^2)^{-1/2}). \tag{SB.171}$$

Collecting the results from (SB.158), (SB.159), (SB.165) and (SB.171) leads to

$$(nT)^{-1}\sum_{t\leq T}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(\bar{u}_{i,\cdot}(w_{i,t}-\bar{w}_{i,\cdot})\varepsilon_{i',t}+\bar{u}_{i',\cdot}(w_{i',t}-\bar{w}_{i',\cdot})\varepsilon_{i,t})=O_p((nT)^{-1/2}). \tag{SB.172}$$

The desired result in (SB.152) follows from (SB.153), (SB.157) and (SB.172). ■

Lemma SB.21 *Under Assumptions 1 and 2, we have:*

$$(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2=\gamma^2\sigma_u^2\hat{\sigma}_c^2+(n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)+O_p(T^{-1/2}).$$

Proof. By Assumptions 1(i, ii, iii), we have

$$\mathbb{E}\left[\left|T^{-1}\sum_{t\le T}(c_t-\bar{c})^2\sum_{i\le n}u_{i,t}\right|^2\right]\le T^{-2}\sum_{t\le T}\mathbb{E}[(c_t-\bar{c})^4]\sum_{i\le n}\mathbb{E}[u_{i,t}^2]\le nT^{-1},$$

which, together with (SA.87) in the proof of Lemma SA.9 and Markov's inequality, shows that

$$\bar{u}T^{-1}\sum_{t\le T}(c_t-\bar{c})^2\sum_{i\le n}u_{i,t}=O_p(T^{-1}). \tag{SB.173}$$

Combining this with Lemma SA.10, (SA.91) in the proof of Lemma SA.11, and (29) in the proof of Lemma 2, we obtain

$$\begin{aligned}(nT)^{-1}\sum_{t\le T}\left((c_t-\bar{c})\sum_{i\le n}(u_{i,t}-\bar{u}_{i,\cdot})\right)^2&=(nT)^{-1}\sum_{t\le T}\left((c_t-\bar{c})\sum_{i\le n}u_{i,t}\right)^2-2\bar{u}T^{-1}\sum_{t\le T}(c_t-\bar{c})^2\sum_{i\le n}u_{i,t}\\&\quad+(nT)^{-1}\sum_{t\le T}\left((c_t-\bar{c})\sum_{i\le n}\bar{u}_{i,\cdot}\right)^2=\sigma_u^2\hat{\sigma}_c^2+O_p(T^{-1/2}).\end{aligned} \tag{SB.174}$$

By Lemma SA.7(ii) and (SA.86) in the proof of Lemma SA.9,

$$\begin{aligned}(nT)^{-1}\sum_{t\le T}(c_t-\bar{c})\left(\sum_{i\le n}(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\le n}\varepsilon_{i,t}\right)&=(nT)^{-1}\sum_{t\le T}(c_t-\bar{c})\left(\sum_{i\le n}u_{i,t}\right)\left(\sum_{i\le n}\varepsilon_{i,t}\right)\\&\quad-\bar{u}T^{-1}\sum_{t\le T}(c_t-\bar{c})\sum_{i\le n}\varepsilon_{i,t}=O_p((nT)^{-1/2}).\end{aligned} \tag{SB.175}$$

Since

$$\begin{aligned}&(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2\\&=\gamma^2(nT)^{-1}\sum_{t\le T}\left((c_t-\bar{c})\sum_{i\le n}(u_{i,t}-\bar{u}_{i,\cdot})\right)^2+(nT)^{-1}\sum_{t\le T}\left(\sum_{i\le n}\varepsilon_{i,t}\right)^2\\&\quad+2\gamma(nT)^{-1}\sum_{t\le T}(c_t-\bar{c})\left(\sum_{i\le n}(u_{i,t}-\bar{u}_{i,\cdot})\right)\left(\sum_{i\le n}\varepsilon_{i,t}\right),\end{aligned}$$

the claim of the lemma follows from Lemma SA.7(i), (SB.174) and (SB.175). ■

Lemma SB.22 *Under Assumptions 1 and 2, we have:*

$$
\begin{aligned}
&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})+\varepsilon_{i,t})\right)^2\\
&=\gamma^2\sigma_u^2(\hat{\sigma}_c^2-(1-T^{-1})\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c})+(n-1)^{-1}(\sigma_u^2\sigma_v^2+\sigma_{u,v}^2)+O_p(T^{-1/2}).
\end{aligned}
\tag{SB.176}
$$

Proof. The term on the left-hand side of the equality in (SB.176) can be expressed as

$$
\begin{aligned}
&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})+\varepsilon_{i,t})\right)^2\\
&=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2+(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi}\right)^2\\
&-2(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)\left(\sum_{i\leq n}(u_{i,t}-\bar{u}_{i,\cdot})(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi}\right).
\end{aligned}
$$

Therefore, by Lemmas SB.18, SB.19 and SB.20, and using (SB.131), we obtain

$$
\begin{aligned}
&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})+\varepsilon_{i,t})\right)^2\\
&=(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}(\gamma(u_{i,t}-\bar{u}_{i,\cdot})(c_t-\bar{c})+\varepsilon_{i,t})\right)^2-\gamma^2\sigma_u^2(1-T^{-1})\hat{\Gamma}_{w,c}^\top\hat{\Sigma}_w^{-1}\hat{\Gamma}_{w,c}+O_p(T^{-1/2}),
\end{aligned}
$$

which, combined with Lemma SB.21, establishes the claim of the lemma. ■

Lemma SB.23 *Under Assumptions 1 and 2, we have*

$$
\begin{aligned}
&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{e,i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi}\right)\right)^2\\
&=(nT)^{-1}\sum_{t\leq T}\left(\begin{array}{c}\sum_{i\leq n}((u_{i,t}-\bar{u}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})+\varepsilon_{i,t})\\ -\gamma(c_t-\bar{c})(\hat{\beta}_{e,iv}-\beta)\sum_{i\leq n}(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})\\ -\sum_{i\leq n}\hat{\phi}_w^\top(w_{i,t}-\bar{w}_{i,\cdot})(\gamma(c_t-\bar{c})-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi})\end{array}\right)^2+O_p((nT)^{-1/2}).
\end{aligned}
$$

Proof. Applying the expression for $z_{i,t}$ in (3), the expression for $x_{i,t}-\bar{x}_{i,\cdot}$ in (SA.1) and the expression for $\hat{u}_{e,i,t}$ in (SB.124), we can express

$$
\sum_{i\leq n}\hat{u}_{e,i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}-(w_{i,t}-\bar{w}_{i,\cdot})^\top\hat{\varphi}\right)
$$

$$
\begin{aligned}
&= \sum_{i \le n} \hat{u}_{e,i,t} \left((n-1)^{-1} \sum_{i' \ne i} \left(x_{i',t} - \bar{x}_{i',\cdot} \right) - (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right) \\
&= \sum_{i \le n} ((u_{i,t} - \bar{u}_{i,\cdot})(\gamma(c_t - \bar{c}) - (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi}) + \varepsilon_{i,t}) - (\hat{\beta}_{e,iv} - \beta)\gamma^2 n (c_t - \bar{c})^2 \\
&- \gamma(c_t - \bar{c}) \sum_{i \le n} \hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot}) + \gamma(c_t - \bar{c})(\hat{\beta}_{e,iv} - \beta) \sum_{i \le n} (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \\
&+ \hat{\phi}_w^\top \sum_{i \le n} (w_{i,t} - \bar{w}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} + I_{1,t} + (\hat{\beta}_{e,iv} - \beta)(I_{3,t} - I_{2,t}) - \hat{\phi}_w^\top I_{4,t}, \qquad \text{(SB.177)}
\end{aligned}
$$

where $\hat{\phi}_w \equiv \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,u} - (\hat{\beta}_{e,iv} - \beta)\hat{\lambda}$, $I_{1,t}$ and $I_{2,t}$ are defined in (SA.71) and (SA.72) respectively. Additionally,

$$
I_{3,t} \equiv \sum_{i \le n} (v_{i,t} - \bar{v}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \quad \text{and} \quad I_{4,t} \equiv (n-1)^{-1} \sum_{i \le n} \sum_{i' \ne i} (w_{i,t} - \bar{w}_{i,\cdot}) \left(v_{i',t} - \bar{v}_{i',\cdot} \right).
$$

By Assumption 1(iii) and Theorem 3, we have

$$
(nT)^{-1} \sum_{t \le T} \left((\hat{\beta}_{e,iv} - \beta)\gamma^2 n (c_t - \bar{c})^2 \right)^2 = O_p(T^{-1}). \qquad \text{(SB.178)}
$$

By Assumptions 1(iii) and 2(iv), and applying Markov's inequality,

$$
(nT)^{-1} \sum_{t \le T} \sum_{i \le n} (c_t - \bar{c})^2 ||w_{i,t} - \bar{w}_{i,\cdot}||^2 = O_p(1). \qquad \text{(SB.179)}
$$

Therefore,

$$
(nT)^{-1} \sum_{t \le T} (c_t - \bar{c})^2 \left(\sum_{i \le n} \hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot}) \right)^2 \le ||\hat{\phi}_w||^2 T^{-1} \sum_{t \le T} (c_t - \bar{c})^2 \sum_{i \le n} ||w_{i,t} - \bar{w}_{i,\cdot}||^2 = O_p(T^{-1}), \qquad \text{(SB.180)}
$$

where the first inequality follows from the Cauchy-Schwarz inequality, and the equality is due to (SB.127) and (SB.179). Similarly, we can show that due to (SB.131)

$$
(\hat{\beta}_{e,iv} - \beta)^2 (nT)^{-1} \sum_{t \le T} \left((c_t - \bar{c}) \sum_{i \le n} (w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right)^2 = O_p(T^{-1}). \qquad \text{(SB.181)}
$$

Applying the Cauchy-Schwarz inequality,

$$
\begin{aligned}
(nT)^{-1} \sum_{t \le T} \left(\sum_{i \le n} \hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right)^2 &\le T^{-1} \sum_{t \le T} \sum_{i \le n} \left(\hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right)^2 \\
&\le T^{-1} ||\hat{\phi}_w||^2 ||\hat{\varphi}||^2 \sum_{t \le T} \sum_{i \le n} \|w_{i,t} - \bar{w}_{i,\cdot}\|^4
\end{aligned}
$$

$$= O_p(T^{-1}), \tag{SB.182}$$

where the last equality is due to Assumption 2(iv), (SB.127) and (SB.131). Therefore, considering Lemma SB.22, (SB.177), (SB.178), (SB.180), (SB.181) and (SB.182), the claim of the lemma follows if

$$\sum_{t\leq T} I_{1,t}^2 + (\hat{\beta}_{e,iv} - \beta)^2 \sum_{t\leq T} (I_{2,t}^2 + I_{3,t}^2) + \sum_{t\leq T} (\hat{\phi}_w^\top I_{4,t})^2 = O_p(1). \tag{SB.183}$$

By similar arguments to those used in proving Lemma SB.18, we can show that

$$(nT)^{-1} \sum_{t\leq T} I_{3,t}^2 = (nT)^{-1} \sum_{t\leq T} \left(\sum_{i\leq n} (v_{i,t} - \bar{v}_{i,\cdot})(w_{i,t} - \bar{w}_{i,\cdot})^\top \hat{\varphi} \right)^2$$
$$= (1 - T^{-1}) \sigma_v^2 \gamma^2 \hat{\Gamma}_{w,c}^\top \hat{\Sigma}_w^{-1} \hat{\Gamma}_{w,c} + O_p(T^{-1/2}) = O_p(1),$$

where the last equality follows from (SB.97) and (SB.99). Combining this with Theorem 3, we have

$$(\hat{\beta}_{e,iv} - \beta)^2 \sum_{t\leq T} I_{3,t}^2 = O_p(1). \tag{SB.184}$$

By Theorem 3 and the same arguments as those used for deriving (SA.78),

$$\sum_{t\leq T} I_{1,t}^2 + (\hat{\beta}_{e,iv} - \beta)^2 \sum_{t\leq T} I_{2,t}^2 = O_p(1). \tag{SB.185}$$

Considering (SB.184) and (SB.185), the desired result in (SB.183) follows if

$$\sum_{t\leq T} (\hat{\phi}_w^\top I_{4,t})^2 = O_p(1). \tag{SB.186}$$

To show (SB.186), we begin by writing

$$\hat{\phi}_w^\top I_{4,t} = (n-1)^{-1} \sum_{i\leq n} \sum_{i'\neq i} \hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot}) \left(v_{i',t} - \bar{v}_{i',\cdot} \right)$$
$$= (n-1)^{-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (\hat{\phi}_w^\top (w_{i,t} - \bar{w}_{i,\cdot}) \left(v_{i',t} - \bar{v}_{i',\cdot} \right) + \hat{\phi}_w^\top (w_{i',t} - \bar{w}_{i',\cdot}) \left(v_{i,t} - \bar{v}_{i,\cdot} \right)). \tag{SB.187}$$

Therefore, by the Cauchy-Schwarz inequality,

$$\sum_{t\leq T} (\hat{\phi}_w^\top I_{4,t})^2 \leq 2||\hat{\phi}_w||^2 \sum_{t\leq T} \left\| (n-1)^{-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (w_{i,t} - \bar{w}_{i,\cdot}) \left(v_{i',t} - \bar{v}_{i',\cdot} \right) \right\|^2$$
$$+ 2||\hat{\phi}_w||^2 \sum_{t\leq T} \left\| (n-1)^{-1} \sum_{i=2}^{n} \sum_{i'=1}^{i-1} (w_{i',t} - \bar{w}_{i',\cdot}) \left(v_{i,t} - \bar{v}_{i,\cdot} \right) \right\|^2. \tag{SB.188}$$

For any $k = 1, \ldots, d_w$, by the Cauchy-Schwarz inequality and Assumptions 1(i, iii) and 2(i, iv), we have

$$\begin{aligned}\mathbb{E}\left[\sum_{t\leq T}\left|\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}^{k}-\bar{w}_{i,\cdot}^{k})\left(v_{i',t}-\bar{v}_{i',\cdot}\right)\right|^{2}\right] &= \sum_{t\leq T}\mathbb{E}\left[\left|\sum_{i=1}^{n-1}(v_{i,t}-\bar{v}_{i,\cdot})\sum_{i'=i+1}^{n}(w_{i',t}^{k}-\bar{w}_{i',\cdot}^{k})\right|^{2}\right] \\ &= \sum_{t\leq T}\sum_{i=1}^{n-1}\mathbb{E}\left[(v_{i,t}-\bar{v}_{i,\cdot})^{2}\left|\sum_{i'=i+1}^{n}(w_{i',t}^{k}-\bar{w}_{i',\cdot}^{k})\right|^{2}\right] \\ &\leq K\sum_{t\leq T}\sum_{i=1}^{n-1}\mathbb{E}\left[(n-i)\sum_{i'=i+1}^{n}(w_{i',t}^{k}-\bar{w}_{i',\cdot}^{k})^{2}\right]\leq Kn^{3}T,\end{aligned}$$

which, together with Markov's inequality, implies that

$$\sum_{t\leq T}\left|(n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i,t}^{k}-\bar{w}_{i,\cdot}^{k})\left(v_{i',t}-\bar{v}_{i',\cdot}\right)\right|^{2}=O_{p}(nT). \tag{SB.189}$$

Similarly, we can show that

$$\sum_{t\leq T}\left|(n-1)^{-1}\sum_{i=2}^{n}\sum_{i'=1}^{i-1}(w_{i',t}^{k}-\bar{w}_{i',\cdot}^{k})\left(v_{i,t}-\bar{v}_{i,\cdot}\right)\right|^{2}=O_{p}(nT).$$

Combining this with (SB.127), (SB.188) and (SB.189) establishes (SB.186). ■

Lemma SB.24 *Suppose that Assumptions 1 and 2 hold. If $T \to \infty$ as $m \to \infty$, then*

$$(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{e,i,t}\left(z_{i,t}-\bar{z}_{i,\cdot}-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)\right)^{2}\to_{p}\gamma^{2}\sigma_{u}^{2}\sigma_{e,c}^{2}+(n_{\infty}-1)^{-1}(\sigma_{u}^{2}\sigma_{v}^{2}+\sigma_{u,v}^{2}).$$

Proof. By (SB.178), (SB.180), (SB.181) and (SB.182), and applying Lemmas SB.22 and SB.23, we have

$$\begin{aligned}&(nT)^{-1}\sum_{t\leq T}\left(\sum_{i\leq n}\hat{u}_{e,i,t}\left((n-1)^{-1}\sum_{i'\neq i}\left(x_{i',t}-\bar{x}_{i',\cdot}\right)-(w_{i,t}-\bar{w}_{i,\cdot})^{\top}\hat{\varphi}\right)\right)^{2} \\ &=\gamma^{2}\sigma_{u}^{2}(\hat{\sigma}_{c}^{2}-(1-T^{-1})\hat{\Gamma}_{w,c}^{\top}\hat{\Sigma}_{w}^{-1}\hat{\Gamma}_{w,c})+(n-1)^{-1}(\sigma_{u}^{2}\sigma_{v}^{2}+\sigma_{u,v}^{2})+O_{p}(T^{-1/2}),\end{aligned}$$

which, along with Assumptions 1(iv) and 2(ii, iii), establishes the claim of the lemma. ■

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